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REMARKS ON SYNTHETIC TABLEAUX FOR CLASSICAL PROPOSITIONAL CALCULUS

Abstract

The Synthetic Tableaux Method of proof for Classical Propositional Calculus is outlined. Basic definitions and theorems (including completeness theorem) are given.

The Synthetic Tableaux Method (STM) is a proof method, which was developed in [6] for Classical Propositional Calculus (CPC) and for some many-valued propositional logics.

In contradistinction to Beth-like semantic tableaux, which are founded on the *reductio ad absurdum* procedure, STM is based on a direct reasoning. STM is constructive in the sense that a proof of a formula A is built up as a set of inferences of A from, defined in a suitable way, consistent sets of propositional variables of A and their negations. STM obeys the subformula principle inasmuch as Beth-like tableaux do: every formula, occurring in a tableau for a formula A must be a subformula of A or a negation of a subformula of A .

In this note we shall outline STM for CPC. We shall also show that STM is an adequate proof method and that it forms a decision procedure for CPC. For the purposes of simplicity we consider CPC with negation and implication as the only primitive connectives (the other connectives can be easily introduced in the standard way). We use p, q, \dots as propositional variables of CPC, φ, ϕ, \dots as metavariables for them, and A, B, \dots as metavariables for formulas of CPC.

Synthetic Inferences

First we shall introduce the notion of a *synthetic inference* of a formula A .

DEFINITION 1. A finite sequence $\mathbf{s} = s_1, \dots, s_n$ of formulas is a *synthetic inference of a formula A* iff:

- (1) for any formula s_i of \mathbf{s} , s_i is a subformula of A or a negation of a subformula of A ;
- (2) s_1 is a propositional variable or a negation of a propositional variable;
- (3) $s_n = A$;
- (4) for any formula s_g of \mathbf{s} , s_g satisfies exactly one of the following conditions:
 - (a) $s_g = \varphi$ (where φ is a propositional variable), and for every f (where $f \neq g$ and $f = 1, \dots, n$), $s_f \neq \varphi$ and $s_f \neq \neg\varphi'$;
 - (b) $s_g = \neg\varphi$ (where φ is a propositional variable), and for every f (where $f \neq g$ and $f = 1, \dots, n$), $s_f \neq \varphi$ and $s_f \neq \neg\varphi'$;
 - (c) s_g is derivable from a certain set of formulas such that each element of this set occurs in \mathbf{s} before s_g .

The derivability relation is determined by the following rules:

$$\text{d1} \frac{A}{\neg\neg A} \quad \text{d2} \frac{B}{(A \rightarrow B)} \quad \text{d3} \frac{\neg A}{(A \rightarrow B)} \quad \text{d4} \frac{A, \neg B}{\neg(A \rightarrow B)}$$

Therefore, a synthetic inference of a formula A is a finite sequence \mathbf{s} of its subformulas or their negations such that the first element of this sequence is a propositional variable or a negation of a propositional variable and the last element is A itself. Moreover, every element of \mathbf{s} is either derivable from some formula(s) occurring earlier in \mathbf{s} or is a propositional variable or a negation of it such that neither this variable nor its negation occur at any other place in \mathbf{s} .

EXAMPLE 1. Consider the following three sequences of formulas:

$$\begin{aligned} \mathbf{g1} &= p, q, \neg\neg q, \neg q \rightarrow \neg p, (p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p) \\ \mathbf{g2} &= p, \neg q, \neg(p \rightarrow q), (p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p) \\ \mathbf{g3} &= \neg p, \neg q \rightarrow \neg p, (p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p) \end{aligned}$$

Each of these sequences is a synthetic inference of the formula $(p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)$.

Synthetic Tableaux

Now we are in a position to introduce our main concept, that is the notion of a *synthetic tableau* for a formula A .

DEFINITION 2. A family Ω of finite sequences of formulas is a *synthetic tableau for a formula A* iff:

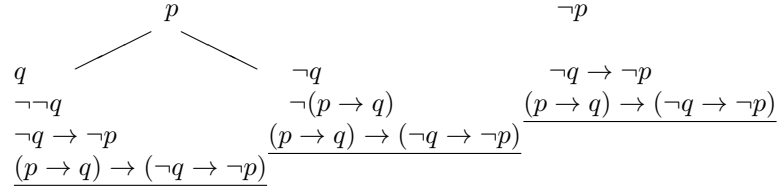
- (1) each element of Ω is a synthetic inference of A or of $'\neg A'$;
- (2) there exists a propositional variable ϕ such that this variable or its negation is the first element of each sequence in Ω ;
- (3) for every sequence $\mathbf{s}=s_1, \dots, s_n$ in Ω the following hold:
 - (a) if $s_i=\phi$ (where $i = 1, \dots, n$), then:
 - (i) Ω contains a certain synthetic inference $\mathbf{s}'=s_1', \dots, s_m'$ such that $s_i=' \neg \phi'$, and, if $i > 1$, then $s_j'=s_j$ for $j=1, \dots, i-1$;
 - (ii) if $i > 1$, then for each synthetic inference $\mathbf{s}''=s_1'', \dots, s_r''$ in Ω such that $s_j''=s_j$ for $j = 1, \dots, i-1$, the following holds: $s_i''=\phi$ or $s_i''=' \neg \phi'$;
 - (b) if $s_i=' \neg \phi'$ (where $i = 1, \dots, n$), then:
 - (i) Ω contains a certain synthetic inference $\mathbf{s}'=s_1', \dots, s_m'$ such that $s_i=\phi$, and, if $i > 1$, then $s_j'=s_j$ for $j=1, \dots, i-1$;
 - (ii) if $i > 1$, then for each synthetic inference $\mathbf{s}''=s_1'', \dots, s_r''$ in Ω such that $s_j''=s_j$ for $j = 1, \dots, i-1$, the following holds: $s_i''=\phi$ or $s_i''=' \neg \phi'$;

Thus a synthetic tableau Ω for a formula A is a set of interconnected synthetic inferences of A or $'\neg A'$ such that every sequence in Ω begins with a fixed propositional variable or its negation. If the i -th element of a sequence \mathbf{s} in Ω is a propositional variable ϕ , then there exists in Ω a certain sequence \mathbf{s}' such that the i -th element of \mathbf{s}' is $'\neg \phi'$ and, if $i > 1$, then \mathbf{s} and \mathbf{s}' do not differ up to their $i-1$ -th elements. Moreover, for every sequence \mathbf{s}'' in Ω such that \mathbf{s} and \mathbf{s}'' do not differ up to their $i-1$ -th elements, the i -th element of \mathbf{s}'' is ϕ or is $'\neg \phi'$. According to the clause (3)(b) of the above definition, analogous conditions are met if the i -th element of a sequence \mathbf{s} in Ω is a negation of a propositional variable.

EXAMPLE 2. The set $\Delta=\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$ of synthetic inferences of the formula $'(p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)'$ (see Example 1) forms a synthetic tableau for that formula.

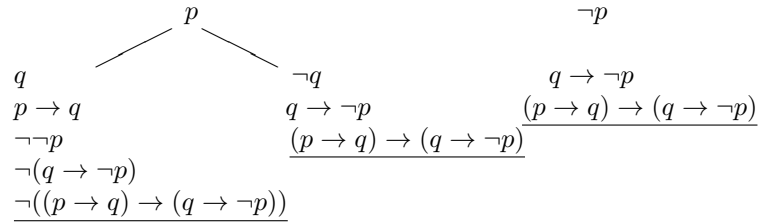
For the purposes of clarity a synthetic tableau for a formula A can be represented in a form of a tree-like diagram.

EXAMPLE 2A. The tableau Δ of Example 2 (represented in a form of a tree-like diagram):



Each branch of the tree consists of the formulas of a certain synthetic inference in Δ : the leftmost one is \mathbf{g}_1 , the rightmost one is \mathbf{g}_3 . By underlining we indicate the last formula of the inference.

EXAMPLE 3. A synthetic tableau for the formula $'(p \rightarrow q) \rightarrow (q \rightarrow \neg p)'$:



Lemmas and Theorems

The following lemmas and theorems can be proved.

LEMMA 1. *A formula A is satisfiable iff there exists a synthetic inference of A .*

SKETCH OF PROOF. A proof of the first part of this lemma shows the way of constructing a synthetic inference for any satisfiable formula. Let w be a valuation such that A is true under it. Let P be a set made up of the subformulas of A or their negations such that the following condition is met:

- (*) for every subformula C of A :
- (1) $C \in P$ iff C is true under w ;
 - (2) $'\neg C' \in P$ iff C is false under w .

Let's then build a sequence \mathbf{s} of the elements of P in the following way:

1. the propositional variables and negations of propositional variables of P precede all the other terms of \mathbf{s} and they are ordered alphabetically (that is, if φ_j and $\neg\varphi'_i$ both occur in P and $i < j$, then $\neg\varphi'_i$ precedes φ_j in \mathbf{s});
2. all the other terms of \mathbf{s} are ordered with respect to increasing degrees of their complexity (defined as the measure of the number of arguments of the connectives); in case of equal degrees of complexity the order is alphabetical;
3. \mathbf{s} is a sequence without repetitions.

It is obvious that the last element of \mathbf{s} is A itself. It can be shown that \mathbf{s} is a synthetic inference of A .

A proof of the second part of lemma 1. is based on the fact that the set of all the formulas of any synthetic inference of a formula A is satisfiable.

LEMMA 2. *For every formula A there exists a synthetic tableau for A .*

SKETCH OF PROOF. Let ϕ_1, \dots, ϕ_k , where $1 < \dots < k$, be all the different propositional variables of A . There exist 2^k different classes of valuations such that valuations belonging to a certain class do not differ with respect to the truth values, assigned to ϕ_1, \dots, ϕ_k . There exist also 2^k k -element sequences without repetitions which terms are propositional variables of A or their negations, ordered alphabetically. In order to define these sequences precisely let first assign to a propositional variable ϕ_j (where $j = 1, \dots, k$) a 2^k -element sequence whose terms are ϕ_j and $\neg\phi'_j$. The f -th term of the sequence \mathbf{u}^j , assigned to the variable ϕ_j is defined as follows:

- (1) $u_f^j = \phi_j$, if $1 \leq f \leq 2^{k-j}$,
- (2) $u_f^j = \neg\phi'_j$, if $2^{k-j} < f \leq 2^{(k-j)+1}$,
- (3) $u_f^j = u_{f-m}^j$, if $2^{(k-j)+1} < f \leq 2^k$, where $m = 2^{(k-j)+1}$.

For every r ($r = 1, \dots, 2^k$), define a k -element sequence \mathbf{v}^r : $\mathbf{v}^r = \langle u_r^1, \dots, u_r^k \rangle$.

For every sequence \mathbf{v}^r there exists a valuation w_n satisfying the following condition:

- (**)
 (1) ϕ_j is true under w_n iff $u_r^j = \phi_j$;
 (2) ϕ_j is false under w_n iff $u_r^j = \neg\phi_j$.

Let w_1, \dots, w_t , where $t = 2^k$ be valuations that met the condition (**) with respect to the sequences $\mathbf{v}^1, \dots, \mathbf{v}^t$. For every valuation w_i ($i = 1, \dots, t$) define a set W_i of propositional variables of A or their negations such that:

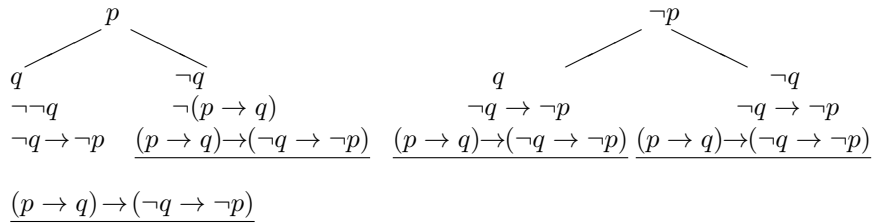
- (1) $\phi_j \in W_i$ iff ϕ_j is true under w_i ;
 (2) $\neg\phi_j \in W_i$ iff ϕ_j is false under w_i .

In this way we obtain 2^k k -element different sequences whose terms are propositional variables of A or their negations.

The proof of lemma 1. shows that there exist 2^k synthetic inferences of A or of $\neg A$ such that alphabetically ordered elements of the sets W_1, \dots, W_t form the k -element initial sequences of these inferences. Moreover, propositional variables of A and their negations occur in these synthetic inferences *only* as terms of their k -element initial sequences. It can be easily shown that these initial sequences are interconnected in a way described by clause (3) of the definition of synthetic tableau (definition 2.) and that a family Ω of all these synthetic inferences of A or of $\neg A$ is a synthetic tableau for the formula A .

By the proofs of lemmata 1. and 2. an effective and mechanical procedure of building a synthetic tableau for any formula is given. Synthetic tableaux constructed in such a way are called *canonical*. Usually, some of the synthetic inferences of a canonical synthetic tableau for a given formula contain some superfluous inferential steps. Consider the following example:

EXAMPLE 4. Canonical synthetic tableau for the formula $(p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)$:



Introduction of the pair $q - \neg q'$ after $\neg p'$ is a superfluous step (cf. example 2a). Mechanical procedure of constructing canonical synthetic tableaux described above forces an introduction into such tableaux synthetic inferences that contain all the possible consistent and "complete" sets of propositional variables of A or their negations. Thus it can be shown that, if k is the number of all the different propositional variables of certain formula A , then the maximal number of synthetic inferences of A or of $\neg A$ sufficient to form a synthetic tableau for this formula is 2^k .

THEOREM 1. *A formula A is a CPC-valid iff there exists a synthetic tableau Ω for A such that every element of Ω is a synthetic inference of A .*

A proof of the completeness part of this theorem is straightforward, in view of the above lemmata. The simplest proof of the soundness part is an indirect one and is based on the notion of *minimal error point* (this notion was introduced by Wisniewski in [7]; for more details concerning its application in STM see [6]).

THEOREM 2. *A formula A is a CPC-valid iff for each synthetic tableau Ω for A any element of Ω is a synthetic inference of A .*

Theorems 1 and 2 show that in order to prove that a formula A is CPC-valid it is sufficient to construct only one synthetic tableau for A and to check whether all of its paths lead to A or not. The fact that STM forms a decision procedure for CPC and enables one to distinguish all the CPC-valid formulas and all the inconsistent formulas can be explicitly expressed by adding to the above theorems the following one:

THEOREM 3. *A formula A is inconsistent iff there exists a synthetic tableau Ω for A such that every element of Ω is a synthetic inference of $\neg A$.*

and a counterpart of Theorem 2., concerning inconsistent formulas.

Closing Remarks

Synthetic tableaux for CPC were originally developed as so-called declarative parts of erotetic search scenarios, that is formal representations of systematic procedures aimed at searching for possible answers to certain

kind of questions (see [7] and chapter IV of [6] for more details). Nevertheless, many of basic intuitions underlying STM are similar to some formal techniques known in the literature. We shall mention three such connections.

L. Kalmar's proof of the completeness of CPC is based on the fact that a CPC-valid formula A can be derived on the basis of every consistent and "complete" set of propositional variables of A or their negations. Kalmar's original method is dependent upon the deductive apparatus chosen, but can be easily generalized.

The automatic proof procedure of Kalish and Montague (see [3]) of deriving formula to be proved from every possible consistent combinations of its propositional variables or their negations is based on the Kalmar's technique. Unfortunately, the authors do not consider the possibility of (quite straightforward, in fact) extending this method into decision procedure.

On the other hand, Dutkiewicz in [2] gives a natural deduction rejection method for CPC (as well as for propositional versions of intuitionistic logic and Feys' system T). His notion of rejection of a formula A corresponds to our notion of synthetic inference of a formula $\neg A$, but Dutkiewicz's method is not automatic and is not aimed at forming a decision procedure. Finally, let us add some remarks concerning the problem of applications of STM. It is not the case that synthetic tableaux method offers a substantially simpler decision procedure for CPC (in terms of computational complexity) than, *e.g.*, resolution method or even Smullyan's analytic tableaux. Nevertheless, STM has some advantages that become visible in case of non-classical logic. One example is given in chapter III of [6]: it is Lukasiewicz calculus L3. Another one is an application of STM to some logics intermediate between CPC and paraconsistent logics (so-called adaptive logics), which leads to elegant and highly intuitive results.

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