Logique & Analyse **173–174–175** (2001), 69–91

SYNTHETIC TABLEAUX AND EROTETIC SEARCH SCENARIOS: EXTENSION AND EXTRACTION

MARIUSZ URBAŃSKI

Abstract
This paper explores the interconnections between synthetic tableaux and erotetic search scenarios.

1. *Introduction*

Synthetic tableaux method (STM) is a model-seeking and proof method. It was developed in detail in Urbański (2001), (2002) and (forthcoming) as a decision procedure for Classical Propositional Calculus (CPC) and for some non-classical logics. Roughly speaking, a synthetic tableau for a formula A is defined as a family of interconnected derivations (so called *synthetic inferences*) of either A or *non-A* (or $\neg A$, in the case of languages with negation), on the basis of all the relevant suitably defined sets of basic constituents of A (in the case of propositional logics they are propositional variables of A) or their negations. The formulas occuring in a synthetic inference of A can only be subformulas of A or their negations.

The fundamental ideas underlying STM can be traced back to L. Kalmár's proof of the completeness of CPC. As the reader will recall, in this proof one makes use of the fact that every valid formula is entailed by every consistent set made up of all of its propositional variables or their negations. However, Kalmár's original proof is system-dependent: it can be applied to every logic which validates certain theorems. STM generalizes its idea. The result is a tableau-style decision procedure which, in contradistinction to Beth-like tableaux, is based on direct reasoning. A STM-proof of a formula A is such a synthetic tableau Ω for A that every element of Ω is a synthetic inference of A. Intuitively it can be said that a formula A is proved if and only if all the possible attempts at "synthetizing" A or *non-A* on the basis of the consistent sets of their subformulas (with the sets of basic constituents of A or their negations interpreted as representing "initial conditions" or "basic assumptions") lead to A. Thus, "one way or another" is the shortest description of the ideas underlying STM as a proof method.

It is extremely natural to extend such a procedure to include what is both usually hidden in formal methods and indispensable in every epistemic activity, that is, questions.

From the perspective of the inferential erotetic logic (IEL; cf., e.g., Wiśniewski (2001)) a synthetic tableau for a given formula A can be conceived as an instruction scheme for a systematic search for solutions to the initial problem expressed by the simple yes-no question: "Is it the case that A?". The procedure involves answering consecutively the operative (auxiliary) questions which are raised by the main question (the initial problem) and the results obtained at previous steps. The basic constituents of A or their negations enter the picture as direct answers to the operative questions. The desired solutions are the direct answers to the main question: A or *non-A*.

The basic intuitions underlying the idea of (one kind of) erotetic search scenarios (cf. Wiśniewski (2001) or (2003)) can be described in a similar way. In the present paper we explore the relations between synthetic tableaux and erotetic search scenarios.

Some of the results presented here (in particular in Section 5) have already appeared in Urbański (2002).

In order to keep the picture as simple as possible, our presentation will be restricted to STM as a proof method for CPC. This somewhat toy-example can easily be extended to cover STM as a model-seeking procedure for other formal systems.

2. *Synthetic tableaux*

Let L be a language of CPC with $\neg$ (negation), $\rightarrow$ (implication), $\wedge$ (conjunction) and $\vee$ (disjunction) as the primitive connectives. We use the symbols p, q, ... as propositional variables of CPC, the symbols ϕ, φ, ... as metavariables for them, the symbols A, B, ... as metavariables for well-formed formulas (or formulas for short) of L and X, Y, ... as variables for sets of wffs of L. We assume that all the basic syntactic and semantic notions of the language L are defined in the standard manner. In particular, by a *valuation* of a set X of formulas of L we mean a mapping which assigns one of the truth values T (Truth), F (Falsehood) to every formula in X. The concept of a *Boolean valuation* is defined in the usual way. By a *compound* formula we mean a formula which is not a propositional variable. We also assume that in the present section all the language-dependent notions are referred to the language L.

We begin with the notion of an atom:

Definition 1: *Every propositional variable and every formula of the form $\neg A$, where A is a propositional variable is an* atom.

Thus, an atom is either a propositional variable, or the negation of a propositional variable. The atoms φ, $\neg\varphi$ will be referred to as *based on* the propositional variable φ. The atoms based on the very same propositional variable will be called *associates*. Thus, an associate to φ is $\neg\varphi$ and vice versa.

Synthetic tableaux are defined as the families of synthetic inferences:

Definition 2: *A finite sequence* $\mathsf{s} = s_1, ..., s_n$ *of formulas is a* synthetic inference of a formula A *iff:*

(1) *for any formula s_i of* s, *s_i is a subformula of A or a negation of a subformula of A;*
(2) *s_1 is an atom;*
(3) *$s_n = A$;*
(4) *for any formula s_g of* s, *s_g satisfies exactly one of the following conditions:*
   (a) *s_g is an atom and the associate to it does not appear in* s*;*
   (b) *s_g is derivable from a certain set of formulas such that each element of this set occurs in* s *before s_g.*

Therefore, a synthetic inference of a formula A is such a finite sequence s of its subformulas or their negations, that the first term of this sequence is an atom and the last term is A itself. Moreover, every term of s is either an atom or is derivable from some formula(s) occurring earlier in s.

The derivability relation is determined by the following rules:

DN-rule
$A/\neg\neg A$

CI1-rule $\neg A/A \to B$	CI2-rule $B/A \to B$	CR-rule $A, \neg B/\neg(A \to B)$
KI-rule $A, B/A \wedge B$	KR1-rule $\neg A/\neg(A \wedge B)$	KR2-rule $\neg B/\neg(A \wedge B)$
DI1-rule $A/A \vee B$	DI2-rule $B/A \vee B$	DR-rule $\neg A, \neg B/\neg(A \vee B)$

It can easily be observed that the derivability relation here is of a "synthetizing" character: the rules describe the way formulas are composed on the basis of their subformulas or their negations.

Example 1.

Consider the following four sequences of formulas:

$\mathsf{c}_1 = p, p \vee q, r, p \wedge r, (p \vee q) \rightarrow r, q, \neg\neg q, \neg((p \wedge r) \rightarrow \neg q), \neg(((p \vee q) \rightarrow r) \rightarrow ((p \wedge r) \rightarrow \neg q))$
$\mathsf{c}_2 = p, p \vee q, r, p \wedge r, (p \vee q) \rightarrow r, \neg q, (p \wedge r) \rightarrow \neg q, ((p \vee q) \rightarrow r) \rightarrow ((p \wedge r) \rightarrow \neg q)$
$\mathsf{c}_3 = p, p \vee q, \neg r, \neg((p \vee q) \rightarrow r), ((p \vee q) \rightarrow r) \rightarrow ((p \wedge r) \rightarrow \neg q)$
$\mathsf{c}_4 = \neg p, \neg(p \wedge r), (p \wedge r) \rightarrow \neg q, ((p \vee q) \rightarrow r) \rightarrow ((p \wedge r) \rightarrow \neg q)$

The sequence c_1 is a synthetic inference of the formula $\neg(((p \vee q) \rightarrow r) \rightarrow ((p \wedge r) \rightarrow \neg q))$ whereas the sequences c_2, c_3, c_4 are synthetic inferences of the formula $((p \vee q) \rightarrow r) \rightarrow ((p \wedge r) \rightarrow \neg q)$.

The main concept of STM is the notion of a synthetic tableau for a given formula:

Definition 3: *A family Ω of finite sequences of formulas is a* synthetic tableau for a formula A *iff:*

(1) *each element of Ω is a synthetic inference of A or of $\neg A$;*
(2) *there exists such a propositional variable φ that the first term of every sequence in Ω is an atom based on φ;*
(3) *for every sequence $\mathsf{s} = s_1, ..., s_n$ in Ω the following holds:*
if s_i is an atom, then:
   (a) *Ω contains a certain synthetic inference $\mathsf{s}' = s'_1, ..., s'_m$ such that s'_i is an associate to s_i and, if $i > 1$, then $s'_j = s_j$ for $j = 1, ..., i-1$;*
   (b) *if $i > 1$, then for each such synthetic inference $\mathsf{s}' = s'_1, ..., s'_r$ in Ω that $s'_j = s_j$ for $j = 1, ..., i-1$, the following holds: $s'_i = s_i$ or s'_i is an associate to s_i.*

Thus, a synthetic tableau Ω for a formula A is a set of interconnected synthetic inferences of A or of $\neg A$ such that every element of Ω begins with an atom based on a fixed propositional variable. If the i-th term of a sequence s in Ω is an atom based on ϕ, then there exists in Ω such a synthetic inference s' that its i-th term is an associate to s_i and, if $i > 1$, then s and s' do not differ up to their i-1th terms. Moreover, for every sequence s' in Ω such that s and s' do not differ up to their i-1th terms the i-th term of s' is an atom based on ϕ.

Example 2.

The set $\Delta = \{\mathsf{c}_1, \mathsf{c}_2, \mathsf{c}_3, \mathsf{c}_4\}$ made up of the synthetic inferences of the Example 1 is a synthetic tableau for the formula $((p \vee q) \to r) \to ((p \wedge r) \to \neg q)$. For clarity it can be presented in a tree-like diagram form in the following way:

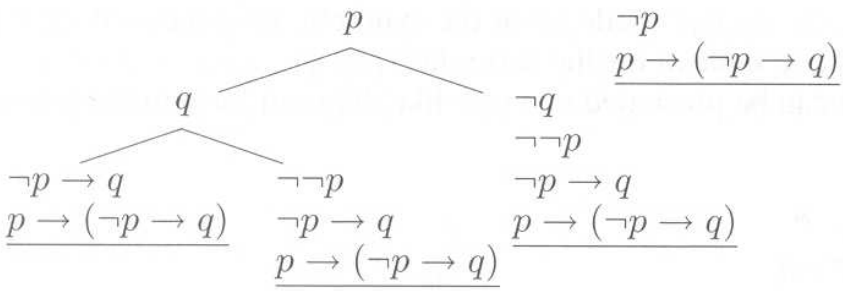
$$
\begin{array}{lll}
 & p & \neg p \\
 & p \vee q & \neg(p \wedge r) \\
 & & (p \wedge r) \to \neg q \\
r & \neg r & \underline{((p \vee q) \to r) \to ((p \wedge r) \to \neg q)} \\
p \wedge r & \neg((p \vee q) \to r) & \\
(p \vee q) \to r & \underline{((p \vee q) \to r) \to ((p \wedge r) \to \neg q)} & \\
 & & \\
q & \neg q & \\
\neg\neg q & (p \wedge r) \to \neg q & \\
\neg((p \wedge r) \to \neg q) & \underline{((p \vee q) \to r) \to ((p \wedge r) \to \neg q)} & \\
\underline{\neg(((p \vee q) \to r) \to ((p \wedge r) \to \neg q))} & &
\end{array}
$$

Every branch of the above tree (or, to be precise, a bush: it is not the case that for a given synthetic tableau Ω there exists such a formula A that every synthetic inference in Ω begins with A; thus there is no trunk in synthetic tableaux) is made up of the formulas of a certain synthetic inference in Ω. The last formula of the inference is indicated by underlining.

It is worth noticing that although synthetic tableaux are defined as sets of synthetic inferences, it is not only convenient to represent a synthetic tableau for a given formula by a tree-like diagram but also to use such diagrams as the genuine form of synthetic tableaux (the reason for this claim will become clear after Section 4).

Now, consider the following synthetic tableau for the formula $p \to (\neg p \to q)$:

Example 3.



Here, the leftmost branch of the tableau and the branch next to it represent two distinct derivations of the formula $p \to (\neg p \to q)$ on the basis of the very same set of atoms. Intuitively, these branches represent two distinct ways of solving the initial problem on the basis of the same evidence (or extralogical assumptions). As a result, the tableau branches not only on pairs of associates, but on the formulas $\neg p \to q$ and $\neg\neg p$, as well. While it is not the case that we would like to avoid such a situation in general, such inferential steps can in a sense be interpreted as superfluous and therefore may be avoided. To this end, we introduce the notion of a regular synthetic tableau, that is a tableau that branches on pairs of associated atoms only.

Definition 4: *A* regular *synthetic tableau for a formula A is a synthetic tableau Ω for A such that for every sequence $\mathsf{s} = s_1, ..., s_n$ in Ω the following conditions hold:*

(1) *if $s_i (i = 1, ..., n)$ is an atom, and $m (i < m < n)$ is an index such that $s_{i+1}, ..., s_{m-1}$ are not atoms and s_m is an atom, then for every sequence s^* in Ω such that $s^*_j = s_j (j = 1, ..., i)$ we have: $s^*_{i+1} = s_{i+1}, ..., s^*_{m-1} = s_{m-1}$;*

(2) *if $s_i (i = 1, ..., n)$ is an atom and for each index $m (i < m \leq n)$ s_m is not an atom, then for every sequence s^* in Ω such that $s^*_j = s_j (j = 1, ..., i)$ we have: $s^*_{i+1} = s_{i+1}, ..., s^*_n = s_n$.*

The tableau of Example 2 is regular, whereas the tableau of Example 3 is not regular.

Some basic properties of STM are given by the following lemmas and theorems:

Lemma 1: *A formula A is satisfiable iff there exists a synthetic inference of A.*

Lemma 2: *For every formula A there exists a synthetic tableau for A.*

Theorem 1: *A formula A is a CPC-valid formula iff there exists such a synthetic tableau Ω for A that every element of Ω is a synthetic inference of A.*

Theorem 2: *A formula A is a CPC-inconsistent formula iff there exists such a synthetic tableau Ω for A that every element of Ω is a synthetic inference of $\neg A$.*

The proofs can be found in Urbański (2001) and (2002).

In the case of decidable systems, an efficient, mechanical and finite procedure for synthetic tableau construction for a given formula can be defined (cf. Urbański (2002) and (forthcoming)). Some remarks concerning the similarities between the basic intuitions underlying STM and other proof methods can be found in Urbański (2001) and (2002).

3. *Erotetic background*

Now we need a language based on the language L of CPC and enriched with questions. Let L^* be the language that results from L by adding to its vocabulary the signs $?, \{, \}$. The set For of formulas of L is the set of declarative formulas of L^* (d-wffs for short). The set $eFor$ of erotetic formulas (that is, questions) of L^* (e-wffs for short) is the smallest set that fulfils the following condition:

(*) if $A_1, ..., A_n$ are distinct d-wffs of L^* (where $n > 1$), then $?\{A_1, ..., A_n\} \in eFor$.

A well-formed formula (wff) of L^* is either a declarative-wff of L^* or a question of L^*. From now on we assume that all the language-dependent notions are referred to the language L^*.

The d-wffs $A_1, ..., A_n$ will be referred to as *direct answers* to the question $?\{A_1, ..., A_n\}$ (for the discussion of the motivations for such a representation of questions cf. Wiśniewski (1995)). We will use the signs Q, Q_1,... as metavariables for questions of L^*. The set of all the direct answers to the question Q will be symbolized by $\mathsf{d}Q$.

According to the condition (*) every set of d-wffs of L^* of at least two elements forms a set of direct answers to some question of L^*. However, some special cases can be distinguished.

A simple yes-no question is a question of the form $?\{A, \neg A\}$. It can be read "Is it the case that A?" and it will be abbreviated as $?A$. A question $?A$ will be referred to as *based* on formula A.

An atomic yes-no question is a simple yes-no question based on a propositional variable. Thus, direct answers to atomic yes-no questions are atoms.

A binary conjunctive question is a question of the form $?\{A \wedge B, A \wedge \neg B, \neg A \wedge B, \neg A \wedge \neg B\}$. It can be read "Is it the case that A and is it the case that B?" and it will be abbreviated as $? \pm \mid A, B \mid$.

The concept of a (generalized) conjunctive question is introduced by the following:

Definition 5: *Let $A_1, ..., A_k (k > 1)$ be distinct declarative formulas. Let $\alpha^{\mathsf{j}}(\mathsf{j} = 1, ..., \mathsf{k})$ be a 2^k-term sequence, whose n-th term is defined in the following way:*

$$\alpha_n^j = \begin{cases} A_j & \text{if } 1 \leq n \leq 2^{k-j} \\ \neg A_j & \text{if } 2^{k-j} < n \leq 2^{(k-j)+1} \\ \alpha_{n-m}^j & \text{if } 2^{(k-j)+1} < n \leq 2^k, \text{ where } m = 2^{(k-j)+1} \end{cases}$$

Let $\beta^{\mathsf{i}}(1 \leq \mathsf{i} \leq 2^k)$ be a k-element sequence, defined in the following way:

$$\beta^{\mathsf{i}} =< \alpha_i^1, \alpha_i^2, ..., \alpha_i^k > .$$

A *conjunctive question with $A_1, ..., A_k$ as its factors* is the question of the form: $?\{C_1, ..., C_t\}$, where $t = 2^k$ and every $C_i (i = 1, ..., t)$ is of the form:

$$(\beta_1^i \wedge (\beta_2^i \wedge ...(\beta_{k-1}^i \wedge \beta_k^i)...).$$

A conjunctive question with $A_1, ..., A_k$ as its factors will be abbreviated as $? \pm \mid A_1, ..., A_k \mid$. Thus a binary conjunctive question $? \pm \mid A, B \mid$ is a conjunctive question with A and B as its factors.

Within the framework of IEL two kinds of erotetic inferences (that is, inferences which have questions as their conclusions) are distinguished. They differ with respect to the premises involved. We are interested here in *valid* erotetic inferences which premises consist of questions and (possibly) of some declarative sentences. The formal counterpart of an intuitive notion of such an inference is the (semantical) concept of the erotetic implication (for discussion concerning the problem of validity of erotetic inferences cf. Wiśniewski (1995)).

Definition 6: (cf. Wiśniewski (1995) or (2001)) *A question* Q implies *a question* Q_1 on the basis of *a set of d-wffs* X *(in symbols:* $\mathsf{Im}(Q, \mathrm{X}, Q_1)$*), iff*

(i) *for each* A *in* $\mathsf{d}Q$*: if all the elements of the set* $\mathrm{X} \cup \{A\}$ *are true under a certain Boolean valuation, then at least one element of the set* $\mathsf{d}Q_1$ *is true under this valuation as well, and*

(ii) *for each* B *in* $\mathsf{d}Q_1$ *there exists a non-empty proper subset* Y *of* $\mathsf{d}Q$ *which fulfils the following condition: if all the elements of the set* $\mathrm{X} \cup \{B\}$ *are true under a certain Boolean valuation, then at least one element of the set* Y *is true under this valuation as well.*

The properties of the erotetic implication are discussed in detail in Wiśniewski (1995). Note, that in the original definition of the concept in question a notion of a multiple-conclusion entailment is used.

In what follows we will make use of the fact that the following (cf. Wiśniewski (2001)) hold:

(1) $\mathsf{Im}(?\pm \mid A_1, ..., A_k \mid, \emptyset, ?A_i)$, where $i = 1, ..., k$.
Thus a conjunctive question Q implies (on the basis of the empty set) every simple yes-no question which is based on a factor of Q.

(2) $\mathsf{Im}(?\neg A, \emptyset, ?A)$.

(3) $\mathsf{Im}(?A\#B, \emptyset, ?\pm \mid A, B \mid)$, where $\#$ is any of the connectives: $\wedge$, $\vee$, $\rightarrow$.

Moreover, one can easily prove the following:

(4) If A contains more than one propositional variable, then $\mathsf{Im}(?A, \emptyset, ?\pm \mid \varphi_1, ..., \varphi_m \mid)$, where $\varphi_1, ..., \varphi_m (m > 1)$ are all the distinct propositional variables of A.
Thus a simple yes-no question based on a formula A implies (on the basis of the empty set) a conjunctive question which factors are all the distinct propositional variables of A.

(5) If A contains only one propositional variable, then $\mathsf{Im}(?A, \emptyset, ?\varphi)$, where φ is the only propositional variable of A.
Thus if A contains only one propositional variable, then the simple yes-no question based on A implies (on the basis of the empty set) the atomic yes-no questions based on this variable.

Erotetic search scenarios are defined as families of erotetic derivations:

Definition 7: (Wiśniewski (2001), (2003)) *A finite sequence* $\mathsf{e} = e_1, ..., e_n$ *of wffs is an* erotetic derivation of a direct answer A to the question Q on the basis of a set of d-wffs X *iff* $e_1 = Q$, $e_k = A$ *and the following hold:*

(1) *for each question* e_k *of* e *such that* $k > 1$*:*
  (a) $\mathsf{d}e_k \neq \mathsf{d}Q$, *and*
  (b) e_{k+1} *is either a question or a direct answer to* e_k*;*

(2) *for each d-wff* e_j *of* e*:*
  (a) $e_j \in \mathrm{X}$, *or*
  (b) e_j *is a direct answer to* e_{j-1}, *where* $e_{j-1} \neq Q$, *or*
  (c) e_j *is entailed by a certain set of d-wffs such that each element of this set precedes* e_j *in* e*;*

(3) *for each question* e_k *of* e *such that* $e_k \neq Q$*:* e_k *is implied by a certain question* e_j *which precedes* e_k *in* e *on the basis of the empty set, or on the basis of a set of d-wffs such that each element of this set precedes* e_k *in* e.

We assume here that entailment is determined by the rules listed in Section 2.

Definition 8: (Wiśniewski (2001), (2003)) *A term* e_k *of an erotetic derivation* $\mathsf{e} = e_1, ..., e_n$ *(where* $1 < k < n$*) is a* query *of* e *iff* e_k *is a question and* e_{k+1} *is a direct answer to* e_k.

Definition 9: (Wiśniewski (2001), (2003)) *A finite family* Φ *of sequences of wffs is an* erotetic search scenario for a question Q relative to a set of d-wffs X *iff each element of* Φ *is an erotetic derivation of a direct answer to* Q *on the basis of* X *and the following conditions hold:*

(1) $\mathsf{d}Q \cap \mathrm{X} = \emptyset$*;*

(2) Φ *contains at least two elements;*

(3) *for each element* $\mathsf{e} = e_1, ..., e_n$ *of* Φ*, for each index* k *such that* $1 \leq k < n$*:*
  (a) *if* e_k *is a question and* e_{k+1} *is a direct answer to* e_k*, then for each direct answer* B *to* e_k*, the family* Φ *contains a certain erotetic derivation* $\mathsf{e}' = e'_1, ..., e'_m$ *such that* $e_j = e'_j$ *for* $j = 1, ..., k$*, and* $e'_{k+1} = B$*;*
  (b) *if* e_k *is a d-wff, or* e_k *is a question and* e_{k+1} *is not a direct answer to* e_k*, then for each erotetic derivation* $\mathsf{e}' = e'_1, ..., e'_m$ *in* Φ *such that* $e_j = e'_j$ *for* $j = 1, ..., k$*, we have* $e'_{k+1} = e_{k+1}$.

Definition 9 neither assumes nor denies that X is a non-empty set. In what follows by an *erotetic search scenario for a question* Q we mean an erotetic search scenario for this question relative to the empty set.

Examples of erotetic derivations and erotetic search scenarios will be given in subsequent sections.

Detailed presentation of erotetic search scenarios can be found in Wiśniewski (2003). For an introduction see also Wiśniewski (2001).

4. *Description of a reasoning*

Now let us focus on the reasoning involved in the construction of the (tree-like diagram of) synthetic tableau for the formula $((p \vee q) \rightarrow r) \rightarrow ((p \wedge r) \rightarrow \neg q)$ (cf. Example 2; for convenience we will call this formula C).

Imagine a systematic inquirer, Mr. I, whose aim is to solve the following (initial) problem (we can call it a "main question"): Is it the case that C or is it the case that $\neg C$? Mr. I has no extralogical background knowledge that could be helpful in this task. Thus, the question becomes: what are the circumstances in which C is the case, and what are the circumstances in which $\neg C$ is the case? His aim is not merely to find out which of the two possibilities holds, but to give a complete description of all the possible cases, that is, to give a kind of conditional schema for approaching the initial problem.

The main question cannot be answered immediately. As a clever logician Mr. I is aware that the answer depends on the combinations of truth-values of the propositional variables of C, thus it is reasonable to pose the relevant question. A question about the truth-values of all the propositional variables of C gives rise to a series of questions about the truth-values of each of these variables. Mr. I starts with the variable p: is p true or is p false? That is: is p the case, or is $\neg p$ the case? If the second possibility holds, then it is quite clear how to proceed in order to solve the initial problem as in this case $\neg(p \wedge r)$ is true and both $(p \wedge r) \rightarrow \neg q$ and C are true as well. But what about the second possibility? Can the truth value of C be determined on the basis of the fact that p is true? Obviously, it is not enough. Thus Mr. I has to seek for new information, that is, information that cannot be derived on the basis of what has been established so far. So he asks one further question: is it the case that r or is it the case that $\neg r$? In the second case the antecedent of C would be false and thus C itself would be true. In the first case (the p-r case, that is, p true, r true) Mr. I can deduce that $p \wedge r$ and the antecedent of C are both true. This is still not enough to answer the main question. Again, Mr. I is in need of some fresh information and again the reasoning has to be split, this time because of the two possible truth-values of q. Mr. I can conclude that on the basis of the assumption that p, r and q are true, the formulas $\neg\neg q$ and $\neg((p \wedge r) \rightarrow \neg q)$ can be derived. As the truth of the antecedent of C has already been established, in this case $\neg C$ holds. On the other hand, if p, r and $\neg q$ are true, both the consequent of C and C itself are true.

What we have described above is a complete scheme for solving Mr. I's initial problem. In principle, we have shown that there is only one possibility of $\neg C$ to be true, that is, the p-q-r case. If any of the other possibilities holds, then C is the case. That solves the initial problem. And Mr. I can conclude his investigations with a general remark that it is not the case that C is true irrespective of the truth-value circumstances (the truth-value distributions over the propositional variables of C), that is, that the general validity of C has been disproved.

In the reasoning described above two kinds of steps were involved. The first kind was of a declarative character: this way Mr. I introduced atoms and formulas obtained by means of the derivability rules. The second kind of steps involved was of an erotetic character: Mr. I posed some (auxiliary) questions, which allowed him to unblock stopped inferences thus governing (in a way) his declarative moves. The justification for the introduction of auxiliary questions was that they were raised by the previous problems and results. As for the purpose for posing them, they were introduced when new information was needed, which could not be obtained simply as a consequence of what had already been established.

If we try to write down Mr. I's reasoning focusing on the declarative steps solely, we will obtain the synthetic tableau for the formula $((p \vee q) \rightarrow r) \rightarrow ((p \wedge r) \rightarrow \neg q)$ of the Example 2. But if we introduce into the picture a representation of the erotetic steps we will obtain a diagram like this:

Example 4.

$?((p \vee q) \rightarrow r) \rightarrow ((p \wedge r) \rightarrow \neg q)$
$? \pm \mid p, q, r \mid$
$?p$

p	$\neg p$
$p \vee q$	$\neg(p \wedge r)$
$?r$	$(p \wedge r) \rightarrow \neg q$
	$((p \vee q) \rightarrow r) \rightarrow ((p \wedge r) \rightarrow \neg q)$

r	$\neg r$
$p \wedge r$	$\neg((p \vee q) \rightarrow r)$
$(p \vee q) \rightarrow r$	$((p \vee q) \rightarrow r) \rightarrow ((p \wedge r) \rightarrow \neg q)$
$?q$	

q	$\neg q$
$\neg\neg q$	$(p \wedge r) \rightarrow \neg q$
$\neg((p \wedge r) \rightarrow \neg q)$	$((p \vee q) \rightarrow r) \rightarrow ((p \wedge r) \rightarrow \neg q)$
$\neg(((p \vee q) \rightarrow r) \rightarrow ((p \wedge r) \rightarrow \neg q))$	

It can easily be seen that the diagram of Example 4 represents an erotetic search scenario for the question $?((p \vee q) \rightarrow r) \rightarrow ((p \wedge r) \rightarrow \neg q)$ relative to the empty set.

5. *Extraction*

Let us now introduce a notion of a declarative part of an erotetic derivation of a certain answer to a given question.

Definition 10: *Let* e *be an erotetic derivation of a direct answer A to the question Q on the basis of a set* X *of declarative wffs. A sequence* s, *formed by elimination of all the erotetic-wffs of* e *is the* declarative part *of* e.

In the Example 5 the declarative part of the leftmost branch of the erotetic search scenario of the Example 4 is shown (for the purposes of clarity this time we write the sequences of formulas downwards):

Example 5.

e = $?((p \vee q) \rightarrow r) \rightarrow ((p \wedge r) \rightarrow \neg q)$
$? \pm \mid p, q, r \mid$
$?p$
p
$p \vee q$
$?r$
r
$p \wedge r$
$(p \vee q) \rightarrow r$
$?q$
q
$\neg\neg q$
$\neg((p \wedge r) \rightarrow \neg q)$
$\neg((p \vee q) \rightarrow r) \rightarrow ((p \wedge r) \rightarrow \neg q)$

s = p
$p \vee q$
r
$p \wedge r$
$(p \vee q) \rightarrow r$
q
$\neg\neg q$
$\neg((p \wedge r) \rightarrow \neg q)$
$\neg((p \vee q) \rightarrow r) \rightarrow ((p \wedge r) \rightarrow \neg q)$

The sequence s is the declarative part of the sequence e.

The notion of a declarative part of an erotetic search scenario for a question Q is given by the following:

Definition 11: *Let* $\Phi = \{\mathsf{e}^1, ..., \mathsf{e}^k\}$ *be an erotetic search scenario for a question* Q *relative to a set* X *of declarative wffs. The* declarative part *of* Φ *is the set* $\Theta = \{\mathsf{s}^1, ..., \mathsf{s}^k\}$, *where* s^i *is a declarative part of* e^i, *for* $i = 1, ..., k$.

The extraction theorem is the following:

Theorem 3: (Extraction Theorem) *Let* $\Phi = \{\mathsf{e}^1, ..., \mathsf{e}^k\}$ *be such an erotetic search scenario for a question* $?A$ *relative to the empty set that:*

(i) *the queries of* Φ *are atomic yes-no questions only;*
(ii) *all the declarative formulas occurring in erotetic derivations of* Φ *are subformulas of* A *or are negations of subformulas of* A*;*
(iii) *erotetic derivations in* Φ *are sequences without repetitions.*

Then there exists such a synthetic tableau Ω for the formula A that Ω is a declarative part of Φ.

Proof. First we show that the declarative part of each erotetic derivation in Φ is a synthetic inference of the formula A or of the formula $\neg A$.
Let e be an erotetic derivation in Φ. Consider the sequence s that is the declarative part of e. Since Φ is a scenario for $?A$ relative to the empty set, then the first declarative term of e (which is at the same moment the first term of s) must be a direct answer to a question occurring in e. As the queries of e are simple yes-no questions only, then s_1 is an atom. Since the last term of every sequence in Φ is A or $\neg A$ (as Φ is a scenario for the question $?A$), therefore the last term of s is A or $\neg A$, respectively. Moreover, every declarative wff B in e (and thus every term of s) meets exactly one of the following conditions:

(1) B is an atom (as a direct answer to a query of e), or
(2) B is entailed by some earlier formulas of e (recall that the entailment here is determined by the rules listed in Section 2).

The sequence e is a sequence without repetitions and so is s. Moreover, this warrants that if an atom occurs in s then its associate does not occur in s. Finally, all the declarative formulas of e (and thus all the terms of s) are subformulas of A or are negations of subformulas of A. Therefore s is a

synthetic inference of A or of $\neg A$.

It is easily visible that the set $\Omega = \{\mathsf{s}^1, ..., \mathsf{s}^k\}$ made up of the declarative parts of all the erotetic derivations in Φ is a synthetic tableau for the formula A. We have already shown that every sequence in Ω is a synthetic inference of A or is a synthetic inference of $\neg A$. The fulfilment of the clauses 3a and 3b of the definition of a synthetic tableau (Definition 2) is guaranteed by the clause 3 of the definition of an erotetic search scenario (Definition 7). ■

As an example of extraction consider the set of sequences that is obtained by the removal of all the erotetic formulas from the erotetic search scenario for the question $?((p \vee q) \rightarrow r) \rightarrow ((p \wedge r) \rightarrow \neg q)$ (cf. Example 3). The declarative part of this scenario is the synthetic tableau for the formula $((p \vee q) \rightarrow r) \rightarrow ((p \wedge r) \rightarrow \neg q)$ (cf. Example 2).

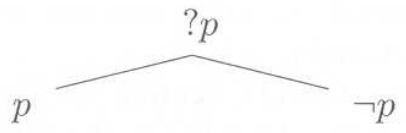
Note, that it is not the case that *every* synthetic tableau is a declarative part of some erotetic search scenario. Let us consider a synthetic tableau for the formula p:

Example 6. (a synthetic tableau for the formula p)

$$\underline{p} \qquad\qquad \underline{\neg p}$$

A scenario of which the Example 6 tableau might be a declarative part should look like this:

Example 7.



However, the two-element family of sequences of Example 7 is not an erotetic search scenario at all, since according to Definition 7 the initial question cannot be a query.

This remark pertains to the atomic yes-no questions only. It can be proved (and we will prove this in the following section) that if A is a compound formula, then for every regular synthetic tableau Ω for A there exists an erotetic search scenario Φ such that Ω is a declarative part of Φ.

6. *Extension*

Lemma 3: *Let A be a compound formula. For each regular synthetic tableau Ω for A there exists an erotetic search scenario Φ for the question $?A$ relative to the empty set such that Ω is the declarative part of Φ.*

Proof. Let A be a compound formula and let Ω be a regular synthetic tableau for A. There are three possibilities: (i) there is only one propositional variable that is a subformula of A, or (ii) A contains more than one propositional variable but the atoms based on only one of them are terms of sequences in Ω, or (iii) A contains more than one propositional variable and atoms based on more than one variable of A occur in Ω.
Assume that the possibility (i) holds. Let φ be the only propositional variable of A. Thus Ω consists of sequences which begin with an atom based on φ and do not contain any other atom. An erotetic search scenario of the required kind is obtained here by adding to every sequence of Ω the initial sequence $?A$, $?\varphi$ (cf. fact (5), Section 3).
Assume that the possibility (ii) holds. Let $\varphi_1, ..., \varphi_m (m > 1)$ be all the distinct propositional variables of A and let φ_k (where $k = 1, ..., m$) be the only variable such that atoms based on it occur as terms of sequences in Ω. An erotetic search scenario of the required kind is obtained here by adding to every sequence of Ω the initial sequence $?A, ?\pm \mid \varphi_1, ..., \varphi_m \mid, ?\varphi_k$ (cf. facts (4) and (1), Section 3).
Assume that the possibility (iii) holds. Let $\varphi_1, ..., \varphi_m (m > 1)$ be all the distinct propositional variables of A. Suppose that the first term of every element of Ω is an atom based on the variable $\varphi_i (i = 1, ..., m)$. First, we add to every element of Ω the following initial sequence $?A, ?\pm \mid \varphi_1, ..., \varphi_m \mid$ (cf. fact (4), Section 3). Second, we stick into any sequence of Ω which contains an atom based on a variable $\varphi_j (j \neq i, j = 1, ..., m)$ the question $?\varphi_j$, placed before the atom (cf. fact (1), Section 3). The result is an erotetic search scenario Φ for the question $?A$ relative to the empty set such that Ω is the declarative part of Φ. ■

As an example of application of the above procedure consider the synthetic tableau for the formula $((p \vee q) \rightarrow r) \rightarrow ((p \wedge r) \rightarrow \neg q)$ (cf. Example 2) and an erotetic search scenario of which it is the declarative part, that is, the scenario for the question $?((p \vee q) \rightarrow r) \rightarrow ((p \wedge r) \rightarrow \neg q)$ (cf. Example 3).

The procedure described in the proof of Lemma 3 is the simplest way of "extension" (regular) synthetic tableaux to erotetic search scenarios but it is not the only one. Although the declarative part of a given erotetic search scenario is uniquely determined, a given (regular) synthetic tableau can be the declarative part of distinct scenarios. For example, this is another erotetic

search scenario of which the synthetic tableau of Example 2 is the declarative part:

Example 8.

$$
\begin{array}{l}
?\,((p \vee q) \rightarrow r) \rightarrow ((p \wedge r) \rightarrow \neg q) \\
?\pm \mid (p \vee q) \rightarrow r, (p \wedge r) \rightarrow \neg q \mid \\
?\,(p \wedge r) \rightarrow \neg q \\
?\pm \mid p \wedge r, \neg q \mid \\
?p \wedge r \\
?\pm \mid p, r \mid \\
?p
\end{array}
$$

$$
\begin{array}{ll}
p & \neg p \\
p \vee q & \neg(p \wedge r) \\
?r & (p \wedge r) \rightarrow \neg q \\
 & ((p \vee q) \rightarrow r) \rightarrow ((p \wedge r) \rightarrow \neg q)
\end{array}
$$

$$
\begin{array}{ll}
r & \neg r \\
p \wedge r & \neg((p \vee q) \rightarrow r) \\
(p \vee q) \rightarrow r & ((p \vee q) \rightarrow r) \rightarrow ((p \wedge r) \rightarrow \neg q) \\
?\neg\neg q & \\
?q &
\end{array}
$$

$$
\begin{array}{ll}
q & \neg q \\
\neg\neg q & (p \wedge r) \rightarrow \neg q \\
\neg((p \wedge r) \rightarrow \neg q) & ((p \vee q) \rightarrow r) \rightarrow ((p \wedge r) \rightarrow \neg q) \\
\neg(((p \vee q) \rightarrow r) \rightarrow ((p \wedge r) \rightarrow \neg q)) &
\end{array}
$$

A more general approach to the problem of extension of synthetic tableaux to erotetic search scenarios is offered by the following definitions:

Definition 12: *Let* s *be a synthetic inference of a formula A. A sequence* e *is an* erotetic extension of s *iff:*

(1) s *is a subsequence of* e*;*
(2) *terms of* s *are the only d-wffs that occur in* e*;*
(3) $e_1 = ?A$*;*
(4) *if an atom based on* ϕ *is a term of* s *and it is the* m*-th term of* e*, then there exists such a sequence of questions* $\mathsf{q} = Q_1, ..., Q_t$ *that:*
  (i) q *is a subsequence of* e*;*
  (ii) $e_{m-1} = ?\phi = Q_t$*;*
  (iii) *for* $h = 1, ..., t$ *such that* $Q_h \neq ?A$ *we have:* $\mathsf{Im}(Q, \mathrm{X}, Q_h)$*, where* Q *is a question that occurs in* e *before* Q_h *and* X *is a*

(possibly empty) set made up of d-wffs such that each element if X *occurs in* e *before* Q_h*;*

(5) *for every questions* e_i, e_j *of* e *(where* $i \neq j$*),* $\mathsf{d}e_i \neq \mathsf{d}e_j$.

By Definitions 12 and 7 it can easily be proved that every erotetic extension of a given synthetic inference of A is an erotetic derivation of the direct answer A to the question $?A$ on the basis of the empty set.

Definition 13: *Let* $\Omega = \{\mathsf{s}^1, ..., \mathsf{s}^k\}$ *be a synthetic tableau for a formula* A. *A set* $F = \{\mathsf{e}^1, ..., \mathsf{e}^k\}$ *is an* erotetic extension of Ω *iff:*

(1) e^i *is an erotetic extension of* s^i*, for every* $\mathsf{i} = 1, ..., \mathsf{k}$*;*

(2) *if for some* $d = 1, ..., k : s_1^d = e_f^d$*, then for every* $i, j = 1, ..., k :$ $e_1^i = e_1^j, ..., e_{f-1}^i = e_{f-1}^j$*;*

(3) *if* s^i *and* s^j *of* Ω *(where* $\mathsf{i} \neq \mathsf{j}$*) do not differ up to their* g*-th terms and their* $g+1$ *terms are associates, then if* $s_{g+1}^i = e_r^i$*, then the sequences* e^i *and* e^j *of* Φ *do not differ up to their* $r-1$ *terms.*

Definitions 12 and 13 do not offer a mechanical procedure for extension of a given synthetic tableau to an erotetic search scenario. However, they give some hints on what such a procedure should look like. First, an erotetic root should be added as an initial subsequence to every synthetic inference in a tableau which is to be extended. Second, every atom occurring in a tableau should be preceded by its erotetic "justification", that is, a sequence of implied questions the last element of which is an atomic yes-no question based on the atom. The condition to be met here is that pairs of associates on which a synthetic tableau branches should be preceded by the very same sequence of questions.

Lemma 4: *Let* A *be a compound formula. Let* Ω *be a regular synthetic tableau for* A. *Each erotetic extension of* Ω *is an erotetic search scenario for the question* $?A$.

Proof. By Definition 9 and Definition 13 ■

Lemma 5: *If* Ω *is a regular synthetic tableau for a compound formula* A, *then there exists at least one erotetic extension of* Ω.

Proof. By Lemma 3 ■

Theorem 4: (Extension Theorem) *Let* A *be a compound formula. Let* Ω *be a regular synthetic tableau for* A. *Then there exists such an erotetic search*

scenario Φ *for the question* $?A$ *relative to the empty set that* Ω *is a declarative part of* Φ.

Proof. By Lemma 4 and Lemma 5 ■

The Extension Theorem is restricted to regular synthetic tableaux only. The reason for such a restriction is a technical one: in the case of non-regular synthetic tableaux augmentation with questions does not produce erotetic search scenarios. Nevertheless, IEL can also shed some light on the reasonings underlying such tableaux. Some parts of them may still be reconstructed in terms of introducing implied questions. In particular, this pertains to the introduction of atoms into a tableau.

7. *Some evaluation*

It is quite obvious that distinct reasonings aimed at solving the very same problem can be evaluated comparatively with respect to some pragmatic factors (in the traditional meaning of the word "pragmatic"). The common label for these factors is "economy of thinking". In comparing the complexity or difficulty of distinct reasonings one can take into account, e.g., the extent of extralogical evidence (or assumptions) that is involved in a given reasoning. The fewer the assumptions the lower the cost of a reasoning (the word "cost" here can be taken very seriously; for instance, consider the problem of planning relevant experiments in natural sciences). The more assumptions there are the greater the possibilities that have to be considered, that is, the number of possible pieces of inferences to be performed.

Another factor that can be taken into account here is the total number of inferential steps (of any kind) involved in a given reasoning. In the case of this factor the relation to the complexity of a reasoning is also a proportional one. Nevertheless, it seems that the first factor is of much greater importance. Usually, the cost of performing logically-based inferences is substantially lower than the cost of obtaining extralogical information.

Now we introduce some notions that can be used in a comparative evaluation of distinct synthetic tableaux with respect to their complexity. These notions are meant to represent the aforementioned complexity-factors. They can be easily adjusted to the evaluation of distinct erotetic search scenarios.

Definition 14:

(a) *The* depth *of a synthetic inference* s *for a formula* A *(in symbols* $\mathrm{d}(\mathsf{s})$*) is the number of atoms, occurring in* s*;*

(b) *The* depth *of a synthetic tableau* Ω *for a formula* A *(in symbols* $\mathrm{d}(\Omega)$*) is the sum of the depths of all the synthetic inferences in* Ω.

Definition 15: *The* extent *of a synthetic tableau* Ω *for a formula* A *(in symbols* $\mathrm{e}(\Omega)$*) is the number of synthetic inferences in* Ω.

Definition 16:

(a) *The* length *of a synthetic inference* s *for a formula* A *(in symbols* $\mathrm{l}(\mathsf{s})$*) is the number of terms of* s*;*

(b) *The* length *of a synthetic tableau* Ω *for a formula* A *(in symbols* $\mathrm{l}(\Omega)$*) is the sum of the lengths of all the synthetic inferences in* Ω.

Our evaluation of distinct synthetic tableaux is basically focused on quantitative factors. What we are going to compare is the relative simplicity of tableaux measured by means of their depth, extent and length. We consider two main criteria of the relative simplicity and an auxiliary one. A synthetic tableau Ω_1 is said to be *relatively simpler* than a synthetic tableau Ω_2 if:

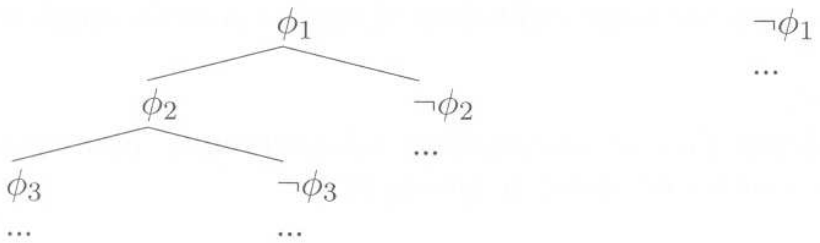
main criteria
(M1) $\mathrm{d}(\Omega_1) < \mathrm{d}(\Omega_2)$;
(M2) $\mathrm{e}(\Omega_1) < \mathrm{e}(\Omega_2)$;

auxiliary criterion
(A) $\mathrm{l}(\Omega_1) < \mathrm{l}(\Omega_2)$.

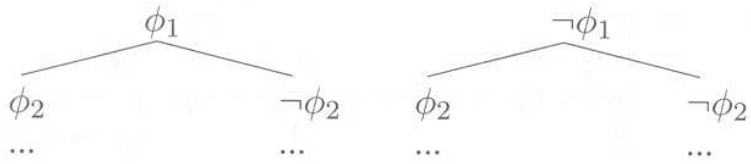
The main criteria measure the "extralogical cost" of reasonings that are represented in evaluated synthetic tableaux (that is, the quantity of information that is introduced into a tableau not by means of the derivability rules). These two criteria are related to each other, but it is not the case that one of them can be reduced to the other. Consider the following two examples of pairs of schemata of synthetic tableaux:

Example 9.

synthetic tableau-scheme Ω_1



synthetic tableau-scheme Ω_2



In the Example 9 we assume that in tableaux Ω_1 and Ω_2 no atoms occur other than those indicated (that is, that there are no other branchings there). Ω_1 and Ω_2 are equal with respect to their extents ($\mathrm{e}(\Omega_1) = \mathrm{e}(\Omega_2) = 4$) but they differ with respect to their depths ($\mathrm{d}(\Omega_1) = 9$ whereas $\mathrm{d}(\Omega_2) = 8$). Thus Ω_2 is relatively simpler than Ω_1.

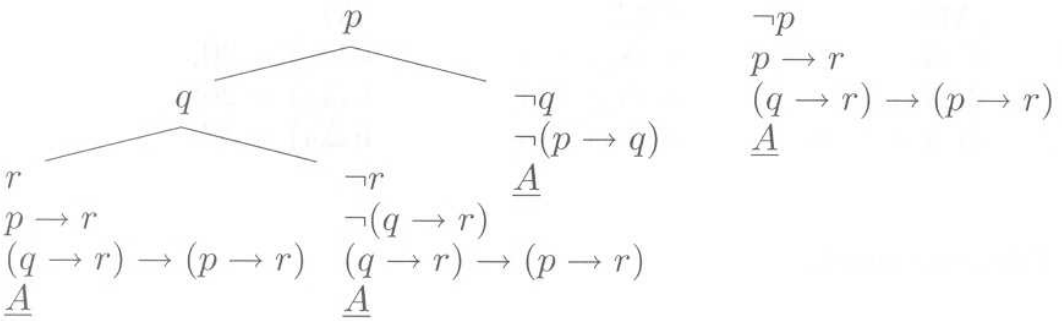
It is possible also that for some tableaux Δ_1, Δ_2, the depth of Δ_1 is greater than the depth of Δ_2, whereas the extent of Δ_2 is greater than that of Δ_1.

The length of a synthetic tableau is the measure of the total number of formulas occurring in the tableau in question. According to what has been said above we take the length of a tableau only as an auxiliary criterion of relative simplicity.

Let us apply our criteria to the evaluation of the relative simplicity of three distinct tableaux for the formula $(p \to q) \to ((q \to r) \to (p \to r))$.

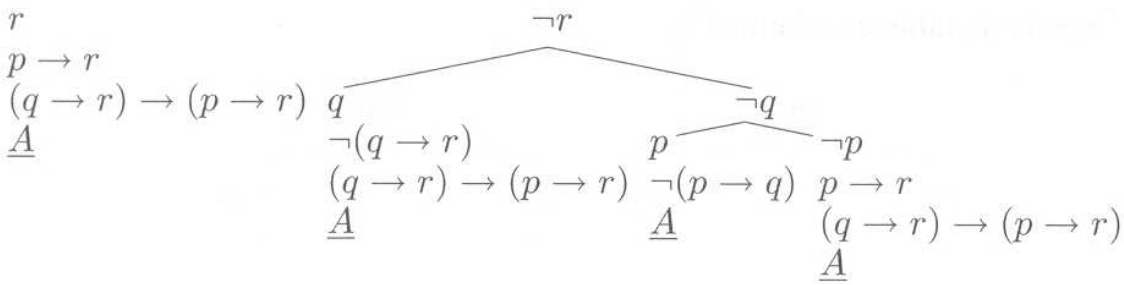
Example 10.

A synthetic tableau Δ_1 for the formula $A = (p \to q) \to ((q \to r) \to (p \to r))$



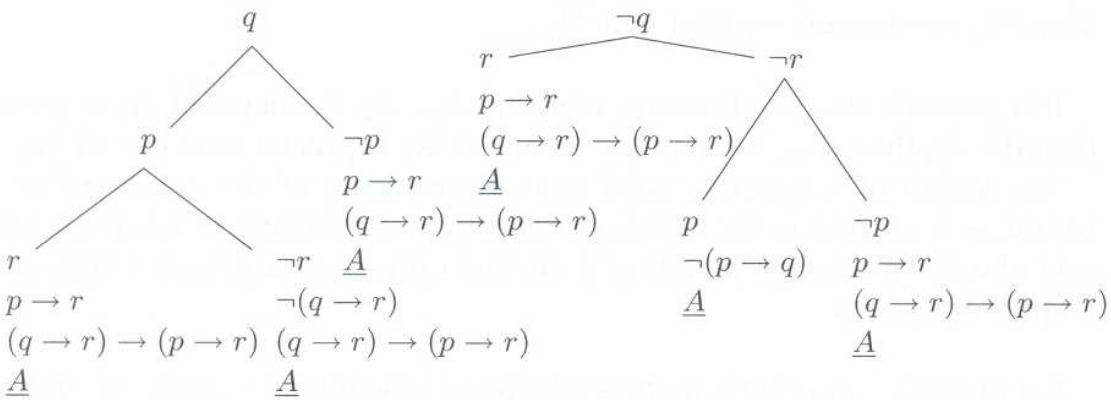
Example 11.

A synthetic tableau Δ_2 for the formula $A = (p \to q) \to ((q \to r) \to (p \to r))$



Example 12.

A synthetic tableau Δ_3 for the formula $A = (p \to q) \to ((q \to r) \to (p \to r))$



At first sight tableaux Δ_1 and Δ_2 do not differ substantially with respect to their relative simplicity whereas tableau Δ_3 is much more complex than the other two. Reckoning the simplicity-factors supports this claim. These are the relevant numbers for the above tableaux:

	(M1)	(M2)	(A)
Δ_1	$\mathrm{d}(\Delta_1) = 9$	$\mathrm{e}(\Delta_1) = 4$	$\mathrm{l}(\Delta_1) = 20$;
Δ_2	$\mathrm{d}(\Delta_2) = 9$	$\mathrm{e}(\Delta_2) = 4$	$\mathrm{l}(\Delta_2) = 20$;
Δ_3	$\mathrm{d}(\Delta_3) = 16$	$\mathrm{e}(\Delta_3) = 6$	$\mathrm{l}(\Delta_3) = 33$.

8. *Closing remarks*

The Extension and Extraction theorems justify the claim that STM is a proof method that is well-grounded in the logic of questions: there are erotetic search scenarios that explicitly represent the "hidden part" of a reasoning involved in establishing a regular synthetic tableau for a given formula. On the one hand, this gives a formal substantiation for some intuitively valid inferential steps performed in STM-style proofs (in particular for the way that atoms are introduced into synthetic inferences). On the other hand the

existence of erotetic extensions of regular synthetic tableaux allows STM to be interpreted as a formal counterpart to the kind of a problem-solving procedure that can be applied to problems expressed by a certain type of questions. An interesting task would be to show that other proof methods and decision procedures (in particular tableaux ones) can also be interpreted in terms of inferential erotetic logic as (declarative) parts of larger cognitive structures. It is possible that a kind of a uniform approach to the formal representation of problem-solving procedures could be established in this way.

Institute of Philosophy
University of Zielona Góra
al. Wojska Polskiego 71a
65-762 Zielona Góra, Poland
E-mail: M.Urbanski@ifil.uz.zgora.pl

REFERENCES

Hunter, G. (1996) *Metalogic*, University of California Press.

Kalish, D. and Montague, R. (1964) *Logic: techniques of formal reasoning*, Harcourt, Brace and World New York.

Robinson, A. and Voronkov, A. (eds.) (2001), *Handbook of Automated Reasoning, vol I–II*, Elsevier Science, Amsterdam.

Urbański, M. (2001) "Remarks on synthetic tableaux for Classical Propositional Calculus", *Bulletin of the Section of Logic* 30, No. 4, pp. 194–204.

Urbański, M. (2002) *Tabele syntetyczne a logika pytań* [*Synthetic Tableaux and the Logic of Questions*], Maria Curie-Skłodowska University Press, Lublin.

Urbański, M. (forthcoming) "Synthetic tableaux for Łukasiewicz's logic Ł3", *Logique et Analyse*.

Wiśniewski, A. (1995) *The Posing of Questions: Logical Foundations of Erotetic Inferences*, Kluwer AP (Synthese Library, vol. 252), Dordrecht.

Wiśniewski, A. (2001) "Questions and Inferences", this volume.

Wiśniewski, A. (2003) "Erotetic search scenarios", *Synthese* 134(3), pp. 389–427.