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SYNTHETIC TABLEAUX FOR ŁUKASIEWICZ'S CALCULUS Ł3

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Abstract

In this paper synthetic tableaux for Łukasiewicz's calculus Ł3 are presented in detail. Basic properties of synthetic tableaux are described as well as a systematic procedure for constructing a tableau for any given formula of Ł3.

1. *Synthetic tableaux*

Synthetic tableaux method (STM) is a semantically-motivated proof method based on direct reasoning[1]. The main idea underlying STM is to solve *via* a tableau the following problem: which (compound) formulas are "synthetizable" (can be derived from the simpler ones) on the basis of certain sets of (atomic) formulas. In the case of Ł3 a synthetic tableau for a formula A is defined as a set of derivations (*synthetic inferences*) of certain expressions, describing truth-functional features of A, on the basis of consistent sets of expressions, describing truth-functional features of propositional variables of A. These expressions, describing the truth-functional features of formulas, will be obtained by the application of the so-called *truth-signs* and will be called *signed formulas* (see section 3).

One of the most distinctive features of STM is that this proof method is well grounded in the logic of questions. Synthetic tableaux were originally developed as so-called declarative parts of erotetic search scenarios (see [8] and [9] for more details), thus they can be interpreted as formal representations of systematic procedures aimed at searching for possible answers to certain kinds of questions.

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[1] For an extended presentation of the STM-method see [6]. The present paper is based on chapter 3 of [6], but is not a mere translation of it.

2. *Basics*

Ł3 is a well-known three-valued propositional calculus, developed by Łukasiewicz (cf. [4] for details). The vocabulary of Ł3 consists of propositional variables (p, q, r, $\ldots$, p_1, p_2, $\ldots$ and so on), implication ($\rightarrow$) and negation ($\neg$) as the only primitive connectives, and brackets as technical symbols (which were originally avoided by Łukasiewicz by using so called *Polish notation*). Ł3 is not truth-functionally complete, but other standard connectives (as conjunction and disjunction) can be defined in it. The notion of a well-formed formula of Ł3 is defined in the standard way. For conciseness, instead of the expression 'well-formed formula of Ł3' we will use the term 'formula' or simply 'wff'. We will use the letters φ, ϕ, $\ldots$ (possibly with subscripts) as metavariables for propositional variables, and the letters A, B, $\ldots$ as metavariables for formulas of Ł3.
The meanings of Ł3 implication and negation are described by the following matrices (in case of implication the first column represents the value of an antecedent, the first row — the value of a consequent):

A	$\neg A$
1	0
$^1/_2$	$^1/_2$
0	1

$\rightarrow$	1	$^1/_2$	0
1	1	$^1/_2$	0
$^1/_2$	1	1	$^1/_2$
0	1	1	1

The sign '1' stands for Truth whereas the sign '0' stands for Falsehood. The symbol '$^1/_2$' stands for undefinedness or undeterminacy (the introduction of the third logical value was strongly connected with Łukasiewicz's indeterministic philosophical account).
A *valuation* is defined as a distribution of the truth-values over the propositional variables. Valuations will be referred to as w, u, $\ldots$ (possibly with subscripts). As 1 is the designated truth value, a *Ł3-valid formula* is a formula which is true under every valuation. The notions of satisfiability of a set of wffs and of a formula are defined in the usual way.

3. *Signed formulas*

Synthetic tableaux for Ł3 will be presented in subsequent sections using so-called *signed* formulas. This concept was introduced by Smullyan (see: [5]) and showed itself as extremely useful in constructing Smullyan-like tableaux. There are two main reasons for that. First, it enables a very concise formulation of inferential rules and a considerable shortening of proofs of metatheorems. The second reason is that for some logics it is impossible to construct a Smullyan-like tableaux without application of signed formulas (see [2] for more details). The same hold for synthetic tableaux.

We will use T, F, N as the truth-signs: if A is a formula, then each of the following: $\mathsf{T}A$, $\mathsf{F}A$, $\mathsf{N}A$ is a *signed* formula. We will use &, #, % as variables for truth-signs.

Truth-signs do not belong to the vocabulary of Ł3, so the truth values of signed formulas are not determined by valuations and matrices for connectives. Nevertheless, the truth value of a signed formula $\#A$ (where # is any of T, F, N) is dependent upon the truth value of the formula A under a certain valuation, so we will speak of the truth value of a formula $\#A$ *with respect* to that valuation. The definition of this notion is given by the following table (in the leftmost column there is indicated the truth value of a formula A under a valuation in question):

A	$\mathsf{T}A$	$\mathsf{F}A$	$\mathsf{N}A$
1	1	0	0
$^1/_2$	$^1/_2$	$^1/_2$	1
0	0	1	0

Thus the truth value of a formula $\mathsf{T}A$ with respect to a certain valuation is the same as the truth value of A under this valuation. The same holds for $\mathsf{F}A$ and $\neg A$. In the third case, that of $\mathsf{N}A$, the situation is slightly different.

In case of Smullyan's analytic tableaux for CPC it is extremely easy to obtain an unsigned version of a tableau from a signed one: it suffices to omit all T's and to transform all F's into '$\neg$'. The same holds in the opposite direction and the same holds for synthetic tableaux for CPC. But this is not the general feature of tableaux: as we have mentioned above, in some cases the truth signs are indispensable. This is also the case here: there is no simple method of elimination of the truth signs in case of synthetic tableaux for Ł3 (at least if one is not going to introduce into the vocabulary of Ł3 some uncommon connectives).

4. *Further syntax*

In what follows we will make use of the notions of a *subformula* of a formula A and of a *degree of complexity* of a formula.
If A is a formula of the form $\neg B$ (or of the form $(B \rightarrow C)$, respectively), then B is (or both B, C are) a *proper subformula(s)* of A and, at the same moment, the only *immediate subformula(s)* of A. If A is a propositional variable, then it has no proper subformulas at all. A formula B is a subformula of a formula A iff (if and only if) B is a proper subformula of A or $A = B$. It follows that if C is a subformula of some subformula of A, then C is a subformula of A. The set of all subformulas of A will be referred to as Sub(A).
The notion of a *degree of complexity* of a formula A (symbolically: deg(A)) is defined here as the measure of the number of arguments of the connectives occurring in A:

(i) if A is a propositional variable, then $\deg(A)$ is 0;

(ii) $\deg(\neg A)$ is $\deg(A) + 1$;

(iii) $\deg(A \rightarrow B)$ is $\deg(A) + \deg(B) + 2$.

Since the truth-signs T, F, N are not symbols of the vocabulary of Ł3, the notions of a subformula and of a degree of complexity of signed formulas are defined in the same way as for their unsigned counterparts:

$$\mathrm{Sub}(A) = \mathrm{Sub}(\mathsf{T}A) = \mathrm{Sub}(\mathsf{F}A) = \mathrm{Sub}(\mathsf{N}A)$$
$$\deg(A) = \deg(\mathsf{T}A) = \deg(\mathsf{F}A) = \deg(\mathsf{N}A)$$

In the construction of the synthetic tableaux for Ł3 the following rules will be applied (we will call them *Ł3-rules*):

CT-rule: $\dfrac{\mathsf{F}A | \mathsf{T}B | \mathsf{N}A, \mathsf{N}B}{\mathsf{T}(A \rightarrow B)}$

CF-rule: $\dfrac{\mathsf{T}A, \mathsf{F}B}{\mathsf{F}(A \rightarrow B)}$

CN-rule: $\dfrac{\mathsf{N}A, \mathsf{F}B | \mathsf{T}A, \mathsf{N}B}{\mathsf{N}(A \rightarrow B)}$

NT-rule: $\dfrac{\mathsf{F}A}{\mathsf{T}\neg A}$

NF-rule: $\dfrac{\mathsf{T}A}{\mathsf{F}\neg A}$

NN-rule: $\dfrac{\mathsf{N}A}{\mathsf{N}\neg A}$

The CT-rule allows one to derive $\mathsf{T}(A \rightarrow B)$ from any of $\mathsf{F}A$, $\mathsf{T}B$ or from the pair $\mathsf{N}A$, $\mathsf{N}B$. The other rules should be understood in an analogous way. We shall call a formula A a *Ł3-consequence* of a set of wffs X iff there exists at least one derivation of A from X by means of the above rules.

In what follows we will omit brackets round arguments of the truth-signs, *e.g.* we will write '$\mathsf{T}A \to B$' instead of '$\mathsf{T}(A \to B)$'. This convention will not lead to any misunderstanding as no signed formula can be a subformula of any wff.

5. *Synthetic inferences*

Definition 1: *Let #A be a signed formula.*
A finite sequence $\mathsf{s} = s_1, \ldots, s_n$ *of signed wffs is a* synthetic inference *of #A iff:*

(1) *every term of* s *is a subformula of A, preceded by a truth-sign;*
(2) s_1 *is a signed propositional variable;*
(3) s_n *is #A;*
(4) *for every* s_g *(where* $g = 1, \ldots, n$*) of* s *one of the following holds:*
   (a) *if* s_g *is a signed propositional variable* $\#\varphi$*, then none of* $\#\varphi$, $\&\varphi$, $\%\varphi$ *(where #, &, % are distinct truth-signs) occurs at any other place in* s*;*
   (b) s_g *is a Ł3-consequence of some earlier formula(s) of* s.

It is obvious that clauses 4a and 4b are disjoint.
Thus a synthetic inference of a signed formula $\#A$ is a finite sequence of signed subformulas of it, which begins with some signed propositional variable and ends with $\#A$ itself. Moreover, every (signed) propositional variable occurs as a term in s only once, no matter of truth-signs, and every formula which is not a (signed) propositional variable is derivable form some earlier formula(s) of s by means of Ł3-rules.
Let us consider two examples:

Example 1
A synthetic inference s_1 of a formula $\mathsf{T}(p \to q) \to (p \to (p \to q))$:
$\mathsf{s}_1 = \mathsf{T}p, \mathsf{N}q, \mathsf{N}p \to q, \mathsf{N}p \to (p \to q), \mathsf{T}(p \to q) \to (p \to (p \to q))$

Example 2
A synthetic inference s_2 of a formula $\mathsf{T}(p \to q) \to (p \to (p \to q))$:
$\mathsf{s}_2 = \mathsf{T}q, \mathsf{T}p \to q, \mathsf{T}p \to (p \to q), \mathsf{T}(p \to q) \to (p \to (p \to q))$

It is easily seen that a synthetic inference of a formula $\#A$ can be viewed as a derivation of $\#A$ on the basis of certain set of signed propositional variables of A such that every propositional variable of A occurs in this set (in a signed form) at most once.
In the above examples the terms of the sequences s_1 and s_2 were ordered

according to their degrees of complexity. It is worthy noticing that such an ordering is not an essential feature of synthetic inferences. As an example, consider the following synthetic inference of the same formula (see also example 6):

Example 3
A synthetic inference s_3 of a formula $\mathsf{T}(p \to q) \to (p \to (p \to q))$:
$\mathsf{s}_3 = \mathsf{F}p, \mathsf{T}p \to q, \mathsf{N}q, \mathsf{T}p \to (p \to q), \mathsf{T}(p \to q) \to (p \to (p \to q))$

Example 3 shows also that synthetic inferences need not be "deductively minimal". That is, they may contain some inferential steps that are deductively superfluous (here this pertains to the formulas $\mathsf{N}q$ and $\mathsf{T}p \to q$).

6. *Synthetic tableaux*

We are now in a position to introduce the main concept of this paper, that is, the notion of a *synthetic tableau* for a given wff of Ł3.

Definition 2: Let A be an unsigned formula. A family Ω of finite sequences of signed wffs is a synthetic tableau *for A iff every element of Ω is a synthetic inference of $\mathsf{T}A$, or of $\mathsf{F}A$, or of $\mathsf{N}A$, there exists a propositional variable φ such that the first element of every sequence in Ω is one of the $\mathsf{T}\varphi$, $\mathsf{F}\varphi$, $\mathsf{N}\varphi$, and for every sequence $\mathsf{s} = s_1, \ldots, s_n$ in Ω the following holds (&, #, % are supposed to represent distinct truth-signs):*

(*) *if s_i (where $i = 1, \ldots, n$) is a signed propositional variable #ϕ, then*
  (1) *there exists in Ω a sequence s^* such that its i-th element is %ϕ and, if $i > 1$, then s and s^* do not differ to the level of their i 1th terms;*
  (2) *there exists in Ω a sequence s^{**} such that its i-th element is &ϕ and, if $i > 1$, then s and s^{**} do not differ to the level of their i 1th terms;*
  (3) *if $i > 1$, then for every sequence s^* in Ω such that s and s^* do not differ to the level of their i 1th terms, the i-th term of s^* is one of the following: $\mathsf{T}\phi$, $\mathsf{F}\phi$, $\mathsf{N}\phi$.*

Thus a synthetic tableau Ω for a formula A is a set of interconnected synthetic inferences of $\mathsf{T}A$, or of $\mathsf{F}A$, or of $\mathsf{N}A$. Every element of Ω begins with a fixed propositional variable, preceded with a truth-sign. If the i-th element of a certain synthetic inference s in Ω is a signed propositional variable #ϕ, then Ω contains synthetic inferences s^*, s^{**} such that they do not differ from s to the level of their i 1th terms and whose i-th terms are %ϕ and &ϕ,

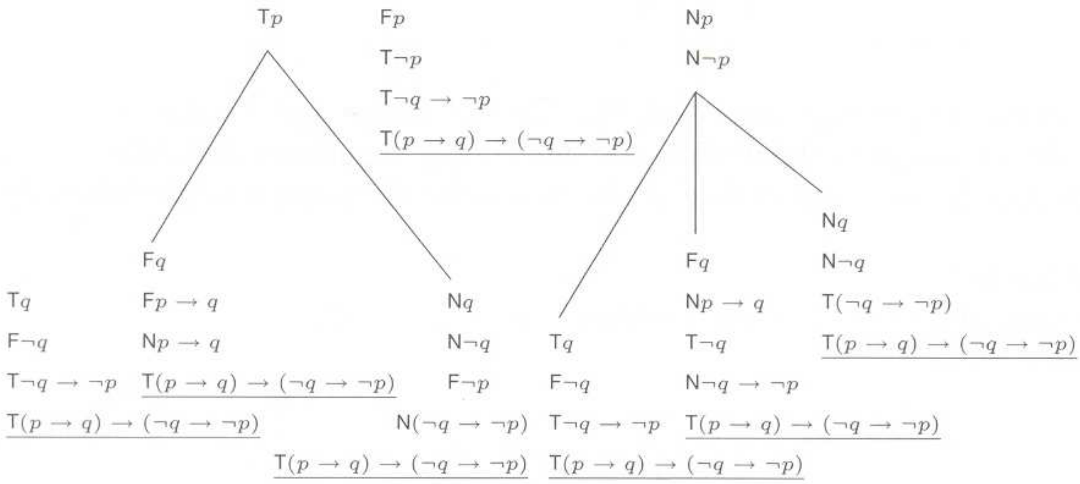
respectively. Moreover, if a synthetic inference s in Ω has a signed propositional variable $\#\phi$ as its i-th term $(i > 1)$, then each synthetic inference in Ω which is identical with s to the level of their i 1th terms has as its i-th term one of the following: $\mathsf{T}\phi$, $\mathsf{F}\phi$, $\mathsf{N}\phi$.

These are two examples of synthetic tableaux for the formulas of Ł3. The first is very simple, the second is a bit more complicated.

Example 4
A synthetic tableau for the formula '$(p \to \neg p) \to \neg p$':

$\mathsf{T}p$	$\mathsf{F}p$	$\mathsf{N}p$
$\mathsf{F}\neg p$	$\mathsf{T}\neg p$	$\mathsf{N}\neg p$
$\mathsf{F}p \to \neg p$	$\underline{\mathsf{T}(p \to \neg p) \to \neg p}$	$\mathsf{T}p \to \neg p$
$\underline{\mathsf{T}(p \to \neg p) \to \neg p}$		$\underline{\mathsf{N}(p \to \neg p) \to \neg p}$

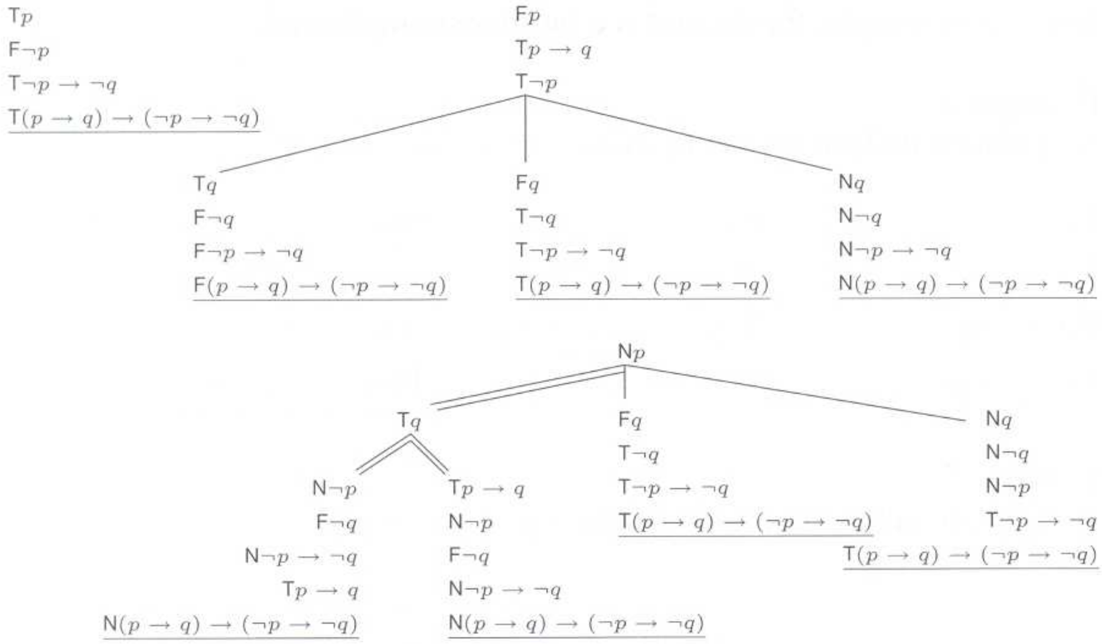
Example 5
A synthetic tableau for the formula '$(p \to q) \to (\neg q \to \neg p)$':



Synthetic tableaux are defined as sets of sequences of wffs. Nevertheless, as in the above examples, we are going to represent them in a tree-like form, where every branch of a tree represents a certain synthetic inference of the tableau in question. The last formula of a synthetic inference is indicated by underlining.
For the purposes of simplicity we shall call the elements of a synthetic tableau Ω *paths* of Ω. If the last element of a path s is $\#A$ we shall say that s *leads* to $\#A$. By T_s we will denote the set of all the (signed) formulas occurring at the path s.

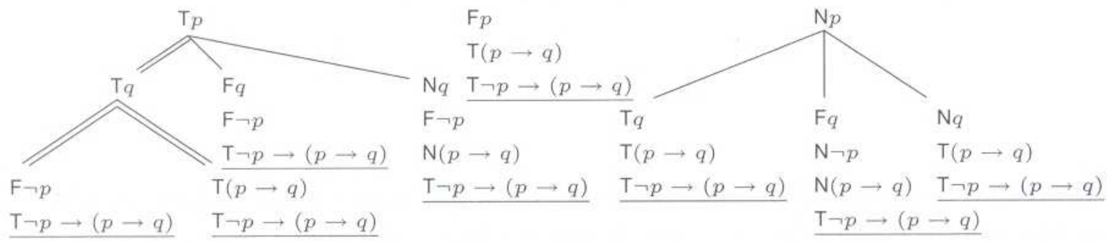
Just as in case of synthetic inferences, "deductive minimality" of synthetic tableaux is not warranted by definition. Consider the following example:

Example 6
A synthetic tableau for the formula '$(p \to q) \to (\neg p \to \neg q)$':



Two inferences beginning with $\mathsf{N}p$, $\mathsf{T}q$ (the fourth and the third one to the right, indicated by the double-line) are simply sequences that differ with respect to the ordering of their terms. A less trivial example is the following:

Example 7
A synthetic tableau for the formula '$\neg p \to (p \to q)$':



In this case the leftmost inference and the inference next to it represent two distinct derivations of the formula '$\mathsf{T}\neg p \to (p \to q)$' on the basis of the very same set of signed variables $\{\mathsf{T}p, \mathsf{T}q\}$.

7. *Minimal error points*

In further metatheorems we will use the notion of a *minimal error point*[2] of a synthetic inference.

Lemma 1: Let a sequence $\mathsf{s} = s_1, \ldots, s_n$ *be a synthetic inference of a certain formula #A and let* w *be a valuation such that* w *does not satisfy* T_s *(recall that* T_s *is the set of all the terms of* s*). Then there exists an index* $k(k = 1, \ldots, n)$ *such that:*

(1) s_k *is a signed propositional variable;*
(2) *the truth value of* s_k *with respect to* w *is* $\mathsf{0}$ *or* $^1/_2$*;*
(3) *there is no* $j < k$ *such that the truth value of* s_j *with respect to* w *is* $\mathsf{0}$ *or* $^1/_2$.

We call k *the* minimal error point *of* s *with respect to* w.

The proof of Lemma 1, which we omit here, is an indirect one and is based on the fact that the Ł3-rules are sound, so the first formula in s which is not true (if any) cannot be a Ł3-consequence of any earlier formulas of s and thus it must be a (signed) propositional variable.

8. *Metatheorems*

In a series of theorems we shall present the basic properties of synthetic tableaux for Ł3.

Lemma 2: Let A *be an unsigned formula and let* w *be a valuation such that* A *is true under* w*. Let W be a set made up of all the propositional variables of* A *such that:*

(*) *for every propositional variable* φ *of* A*:*
  (1) $\mathsf{T}\varphi$ *belongs to W iff* φ *is true under* w*;*
  (2) $\mathsf{F}\varphi$ *belongs to W iff* φ *is false under* w*;*
  (3) $\mathsf{N}\varphi$ *belongs to W iff* φ *is undetermined under* w.

Then $\mathsf{T}A$ *is a Ł3-consequence of W and there exists a synthetic inference of* $\mathsf{T}A$.

Proof
First we shall show that $\mathsf{T}A$ is a Ł3-consequence of W, that is, that there exists a derivation of $\mathsf{T}A$ from W by means of Ł3-rules.

[2] This notion is due to Wiśniewski (cf. [9]).

Let P be the set of all the subformulas of A, preceded by a truth-sign. Let P* be the smallest subset of P such that:

(**) for every subformula C of A:
  (1) $\mathsf{T}C$ belongs to P* iff C is true under w;
  (2) $\mathsf{F}C$ belongs to P* iff C is false under w;
  (3) $\mathsf{N}C$ belongs to P* iff C is undetermined under w.

It is obvious that W is a subset of P* and that $\mathsf{T}A$ belongs to P*.
Now, build up a sequence $\mathsf{d} = d_1, \ldots, d_s$ of the elements of P* ordered by the increasing degree of their complexity in such a way that every formula of degree n should precede in d any formula of degree $n+1$. An exact order of the formulas of the same degree does not matter, with two exceptions:

(i) signed propositional variables should be ordered alphabetically;
(ii) the last element of d should be $\mathsf{T}A$ (this clause is added for the purpose of clarity only; in fact it is superfluous).

Moreover, d should be a sequence without repetitions.
Thus, if $\varphi_1, \ldots, \varphi_t$ are all the propositional variables of A, then the t first terms of d are signed propositional variables and the degree of complexity of any further term of d is greater than 0. It can be shown by induction on the degree of complexity of the terms of d that for any $i (i = 1, \ldots, s)$, d_i is a Ł3-consequence of W:

(I) If $\deg(d_i) = 0$, then d_i is a signed propositional variable and, at the same moment, d_i is in W;
(II) Suppose that $\deg(d_i) > 0$; moreover, suppose that for any $m < i$, d_m is a Ł3-consequence of W.
Obviously, if $\deg(d_i) > 0$, then there is a formula G such that d_i is in either of the form $\mathsf{T}\neg G$ or $\mathsf{F}\neg G$ or $\mathsf{N}\neg G$, or there are formulas G, H such that d_i is in either of the form $\mathsf{T}G \rightarrow H$ or $\mathsf{F}G \rightarrow H$ or $\mathsf{N}G \rightarrow H$.
  (a) If d_i is $\mathsf{T}\neg G$, then, since $\neg G$ is a subformula of A, then G is a subformula of A as well. As $\mathsf{T}\neg G$ is member of P*, $\neg G$ is true under w and G is false under w. This means that $\mathsf{F}G$ is in P* and that there exists an index f such that d_f is $\mathsf{F}G$. Since $\deg(\mathsf{F}G) < \deg(\mathsf{T}\neg G)$, then $f < i$. Thus, on the basis of inductive assumption and by the NT-rule (and due to idempotency of the consequence operator in question), d_i is a Ł3-consequence of W.

The other cases can be proved in a similar way.

This ends the first part of the proof.
It is easily seen that the sequence d is also a synthetic inference of the formula $\mathsf{T}A$. For observe the following:

(1) every term of d is a signed subformula of A;
(2) the first term of d is a signed propositional variable;
(3) the last term of d is $\mathsf{T}A$;
(4) every term of d is either a signed propositional variable (and, due to the conditions imposed on the set W and the sequence d, clause 4a of definition 1 is met) or is a Ł3-consequence of some earlier formula(s) of d. ■

The following two lemmas can be proved in an analogous way:

Lemma 3: *Let A be an unsigned formula and let w be a valuation such that A is false under w. Let W be a set made up of all the propositional variables of A such that:*

(*) *for every propositional variable φ of A:*
  (1) $\mathsf{T}\varphi$ *belongs to W iff φ is true under w;*
  (2) $\mathsf{F}\varphi$ *belongs to W iff φ is false under w;*
  (3) $\mathsf{N}\varphi$ *belongs to W iff φ is undetermined under w.*

Then $\mathsf{F}A$ is a Ł3-consequence of W and there exists a synthetic inference of $\mathsf{F}A$.

Lemma 4: *Let A be an unsigned formula and let w be a valuation such that A is undetermined under w. Let W be a set made up of all the propositional variables of A such that:*

(*) *for every propositional variable φ of A:*
  (1) $\mathsf{T}\varphi$ *belongs to W iff φ is true under w;*
  (2) $\mathsf{F}\varphi$ *belongs to W iff φ is false under w;*
  (3) $\mathsf{N}\varphi$ *belongs to W iff φ is undetermined under w.*

Then $\mathsf{N}A$ is a Ł3-consequence of W and there exists a synthetic inference of $\mathsf{N}A$.

The central theorems of the present paper are the following soundness and completeness theorems:

Theorem 1: *If a formula A is Ł3-valid, then there exists a synthetic tableau Ω for A such that every path of Ω leads to $\mathsf{T}A$.*

Proof

Let A be a certain valid formula of Ł3. Let K be the set of all the propositional variables of A. Let $\varphi_1, \ldots, \varphi_p$ be all the distinct members of K, ordered alphabetically. Thus there exist $t = 3^p$ valuations, which differ with respect to truth values assigned to the variables $\varphi_1, \ldots, \varphi_p$. Let's now assign to every propositional variable $\varphi_h (h = 1, \ldots, p)$ t-terms sequence u^{h} of the truth values 1, $^1/_2$, 0; the g-th $(g = 1, \ldots, t)$ term of the sequence u^h is defined in the following way ('$\times$' stands here for the sign of multiplication):

$$u_g^h = \begin{cases} 1 & \text{if } 1 \leq g \leq 3^{p-h} \\ 0 & \text{if } 3^{p-h} < g \leq 2 \times 3^{p-h} \\ ^1/_2 & \text{if } 2 \times 3^{p-h} < g \leq 3^{(p-h)+1} \\ u_{g-r}^h & \text{if } 3^{(p-h)+1} < g \leq 3^p\text{, where } r = 3^{(p-h)+1} \end{cases}$$

Now, define $t = 3^p$ p-terms sequences of truth values; the sequence v^z (where $z = 1, \ldots, t$) is given by the following formula:

$$\mathsf{v}^z = u_z^1, u_z^2, \ldots, u_z^p$$

Every sequence v^z defines a certain class of valuations, namely, these under which the variable φ_h takes the value u_z^h for $h = 1, \ldots, p$. Moreover, every valuation belongs to exactly one such a class. Valuations which belong to distinct classes (that is, defined by distinct sequences $\mathsf{v}^1, \ldots, \mathsf{v}^t$) do differ with respect to the truth values assigned to the variables of A (that is, $\varphi_1, \ldots, \varphi_p$). Let $w^1, \ldots, w^t$ be valuations which belong to the classes determined by $\mathsf{v}^1, \ldots, \mathsf{v}^t$, respectively.

Now for every valuation w^b (where $b = 1, \ldots, t$) we define a p-term sequence $\mathsf{d}^b = d_1^b, \ldots, d_p^b$ of signed propositional variables of A. The j-th term of the sequence d^b is defined in the following way:

$$d_j^b = \begin{cases} \mathsf{T}\varphi_j & \text{if } \varphi_j \text{ is true under } w^b; \\ \mathsf{F}\varphi_j & \text{if } \varphi_j \text{ is false under } w^b; \\ \mathsf{N}\varphi_j & \text{if } \varphi_j \text{ is undetermined under } w^b \end{cases}$$

where $j = 1, \ldots, p$.[3]

Note, that d^b contains no repetitions.

[3] Consider the following example. Let $A =$ '$p \wedge q \rightarrow p$'. Thus all the distinct variables of A are p, q (ordered alphabetically). The sequence u^p of truth-values, assigned to the variable p will look as follows: $\mathsf{u}^p = 1, 1, 1, 0, 0, 0, ^1/_2, ^1/_2, ^1/_2$ whereas to the variable q will be assigned the sequence $\mathsf{u}^q = 1, 0, ^1/_2, 1, 0, ^1/_2, 1, 0, ^1/_2$. The relevant v-sequences are: $\mathsf{v}^1 = 1, 1$, $\mathsf{v}^2 = 1, 0$, $\mathsf{v}^3 = 1, ^1/_2$, $\mathsf{v}^4 = 0, 1$, $\mathsf{v}^5 = 0, 0$, $\mathsf{v}^6 = 0, ^1/_2$, $\mathsf{v}^7 = ^1/_2, 1$, $\mathsf{v}^8 = ^1/_2, 0$, $\mathsf{v}^9 = ^1/_2, ^1/_2$. As for an example of d-sequences: the sequence determined by a valuation that is defined by the sequence v^3 is the following: $\mathsf{d}^3 = \mathsf{T}p, \mathsf{N}q$.

Let W_b be the set of all the formulas of d^{b}. The following condition is met by W_b:

(*) for every propositional variable φ of A:

(1) $\mathsf{T}\varphi$ belongs to W_b if φ is true under w;

(2) $\mathsf{F}\varphi$ belongs to W_b if φ is false under w;

(3) $\mathsf{N}\varphi$ belongs to W_b if φ is undetermined under w.

Since A is Ł3-valid, then it is true under every valuation, including $w^1, \ldots, w^t$. Thus, by Lemma 2, for any of $w^1, \ldots, w^t$ there exists a certain Ł3-derivation of $\mathsf{T}A$ on the basis of $W_1, \ldots, W_t$, respectively. Now we assign to every of $W_1, \ldots, W_t$ such a derivation, built up as in the proof of Lemma 2; the only restriction is that derivation D^{b} assigned to the set W_b begins with d^{b} as its starting sequence, that is, $D^b_e = d^b_e$ for $e = 1, \ldots, p$ (according to the definition of the sequence d^{b} signed propositional variables of A occur in it in the alphabetical order, determined by their unsigned counterparts). By Lemma 2, any of the derivations $\mathsf{D}^1, \ldots, \mathsf{D}^{\mathsf{t}}$ is a synthetic inference of $\mathsf{T}A$ (and none of them are identical).

Let $\Omega = \{\mathsf{D}^1, \ldots, \mathsf{D}^{\mathsf{t}}\}$. We shall show that Ω is a synthetic tableau for the formula A.

(1) Every sequence in Ω is a synthetic inference of $\mathsf{T}A$.

(2) The first term of every sequence in Ω is the signed propositional variable φ_1 and at least one sequence in Ω begins with $\mathsf{T}\varphi_1$, at least one with $\mathsf{F}\varphi_1$ and at least one with $\mathsf{N}\varphi_1$.

(3) The first p terms of every sequence in Ω are signed propositional variables of A, ordered alphabetically. Thus, if #, &, % are distinct truth-signs, then if $\#\phi$ is the q-th term of the sequence D^{s} (where $s = 1, \ldots, t$), then Ω contains sequences $\mathsf{D}^{\mathsf{s}^*}$, $\mathsf{D}^{\mathsf{s}^{**}}$ which do not differ from D^{s} up to the level of their q 1th terms (if there are any) and are such that the q-th term of $\mathsf{D}^{\mathsf{s}^*}$ is $\&\phi$ while the q-th term of $\mathsf{D}^{\mathsf{s}^{**}}$ is $\%\phi$. Moreover, if $\#\phi$ is the q-th term of D^{s}, then there is no term in D^{s} of any of the form: $\mathsf{T}\phi$, $\mathsf{F}\phi$, $\mathsf{N}\phi$ other than the q-th. Finally, if D^{s} and $\mathsf{D}^{\mathsf{s}^*}$ do not differ up to their q 1th terms and D^s_q is $\#\phi$, then $D^{s^*}_q$ is in one of the forms: $\mathsf{T}\phi$, $\mathsf{F}\phi$, $\mathsf{N}\phi$.

Thus Ω is a synthetic tableau for A such that every path of it leads to $\mathsf{T}A$. ■

Theorem 2: If there exists a synthetic tableau Ω for a formula A such that every path of Ω leads to $\mathsf{T}A$, then A is Ł3-valid.

Proof[4]

Suppose that there exists a synthetic tableau Ω for a formula A such that every path of Ω leads to $\mathsf{T}A$. Moreover, suppose (for an indirect proof) that A is not Ł3-valid. Since all the terms of paths of Ω are signed subformulas of A, Ω is a finite set.

Since A is not Ł3-valid, there exists a certain valuation w such that A is false under w or A is undetermined under w, that is, either $\mathsf{F}A$ or $\mathsf{N}A$ is true with respect to w. Since every path of Ω leads to $\mathsf{T}A$, then w does not satisfy any set of formulas of a path of Ω. Therefore, by Lemma 1, for every path of Ω there exists a minimal error point with respect to the valuation w. Let ω be the set of all the minimal error points of the paths of Ω with respect to the valuation w (so, ω is a set of indices, *i.e.* positive integers).

The set ω is finite and thus must have a maximal element, that is, there exists an index k such that for any j in ω, $j \leq k$. Let s be a path of Ω such that its minimal error point is the maximal one, that is, k. By Lemma 1, s_k is a signed propositional variable, s_k is false or is undetermined with respect to w and either s_k is the first term of s, or every term that precedes s_k in s is true with respect to w.

Let #, &, % be the distinct truth-signs. If s_k is the first term of s, then there are paths s^*, s^{**} of Ω such that, if $s_k = \#\phi$, then $s_k^* = \&\phi$ and $s_k^{**} = \%\phi$. Therefore, one of the s_k^*, s_k^{**} is bound to be true with respect to w.

If s_k is not the first term of s, then there exist paths s^*, s^{**} of Ω such that for every s_d which precedes s_k in s, $s_d = s_d^* = s_d^{**}$ (remember that every term of s which precedes s_k is true with respect to the valuation w) and, if $s_k = \#\phi$, then $s_k^* = \&\phi$ and $s_k^{**} = \%\phi$. Therefore, one of the s_k^*, s_k^{**} is bound to be true with respect to w.

By Lemma 1, for every path of Ω there exists a minimal error point with respect to the valuation w, so there exist also such minimal error points of the paths s_k^* and s_k^{**}. According to what has been shown above, the minimal error point of one of them must be greater than the minimal error point of s. This means that the minimal error point of either s^* or s^{**} is greater than the maximal element of ω, so we arrive at a contradiction. Therefore the valuation w satisfies the set of all the formulas of at least one path of Ω. Since every path of Ω leads to $\mathsf{T}A$, then $\mathsf{T}A$ is true with respect to w and thus A itself is true under w. ■

By using similar ideas as in the proof of Theorem 1, one can prove the following:

[4] This proof is based on the idea that comes from Wiśniewski's [9].

Theorem 3: *For every formula A of Ł3 there exists a synthetic tableau for A.*

One can also prove the following:

Theorem 4: *A formula A is Ł3-valid iff every synthetic tableau Ω for A is such that every path of Ω leads to* $\mathsf{T}A$.

Therefore, one can show that synthetic tableaux method is both an adequate proof method and an effective decision procedure for Ł3. Moreover, the situation is similar in the case of any propositional Łukasiewicz-like finite-valued calculus.

9. *A systematic procedure*

In the present subsection we describe a systematic, finite and effective procedure for constructing a synthetic tableau for any formula of Ł3 that is based on the proofs of lemmata 2–4 and of Theorem 1.

To this end we need to determine an alphabetical order of the formulas of Ł3. Suppose that propositional variables of Ł3 are made up of the symbols φ and $|$: *e.g.* the fourth variable becomes $\varphi\ ||||$. Thus the alphabet of Ł3 consists of the following six signs:

$$\neg \rightarrow ()\varphi\ |$$

We shall call this ordering *alphabetical.* An alphabetical order of the formulas of Ł3 with respect to the alphabetical ordering of the symbols of Ł3 is defined in the standard way.
Now we are in a position to describe a systematic procedure for constructing a synthetic tableau for any formula of Ł3.

Let $\varphi_1, \ldots, \varphi_k$ be all the distinct propositional variables of a formula A of Ł3.

Task
Build a synthetic tableau for A.

Step 1
Determine all the valuations that differ with respect to the truth-values assigned to variables $\varphi_1, \ldots, \varphi_k$. Let $w_1, \ldots, w_r$ be all this distinct valuations[5].

Step 2
Perform the following substeps for w_1:

Substep 2a
Determine the truth value of A under w_1.

Substep 2b
If A is true under w_1, then build up a set W_1 of all the formulas that fulfil the following condition:
(*) for every propositional variable φ_i of A ($i = 1, \ldots, k$):
(1) $\mathsf{T}\varphi_i \in \mathrm{W}_1$ iff φ_i is true under w_1,
(2) $\mathsf{F}\varphi_i \in \mathrm{W}_1$ iff φ_i is false under w_1,
(3) $\mathsf{N}\varphi_i \in \mathrm{W}_1$ iff φ_i is undetermined under w_1,
and then build a synthetic inference s^1 of $\mathsf{T}A$ as in the proof of Lemma 2. If in s^1 there occur some formulas with equal degrees of complexity, they should be ordered alphabetically.

Substep 2c
If A is false under w_1, then build up a set W_1 of all the formulas that fulfil the following condition:
(*) for every propositional variable φ_i of A ($i = 1, \ldots, k$):
(4) $\mathsf{T}\varphi_i \in \mathrm{W}_1$ iff φ_i is true under w_1,
(5) $\mathsf{F}\varphi_i \in \mathrm{W}_1$ iff φ_i is false under w_1,
(6) $\mathsf{N}\varphi_i \in \mathrm{W}_1$ iff φ_i is undetermined under w_1,
and then build a synthetic inference s^1 of $\mathsf{F}A$ as in the proof of Lemma 3. If in s^1 there occur some formulas with equal degrees of complexity, they should be ordered alphabetically.

[5] To be precise, $w_1, \ldots, w_r$ represent distinct classes of valuations such that elements of the same class do not differ with respect to the truth-values assigned to variables $\varphi_1, \ldots, \varphi_k$.

Substep 2d

If A is undetermined under w_1, then build up a set W_1 of all the formulas that fulfil the following condition:

(*) for every propositional variable φ_i of A $(i = 1, \ldots, k)$:

(7) $\mathsf{T}\varphi_i \in \mathrm{W}_1$ iff φ_i is true under w_1,

(8) $\mathsf{F}\varphi_i \in \mathrm{W}_1$ iff φ_i is false under w_1,

(9) $\mathsf{N}\varphi_i \in \mathrm{W}_1$ iff φ_i is undetermined under w_1,

and then build a synthetic inference s^1 of $\mathsf{N}A$ as in the proof of Lemma 4. If in s^1 there occur some formulas with equal degrees of complexity, they should be ordered alphabetically.

Step 3

Perform the substeps 2a–2d for the valuations $w_2, \ldots, w_r$.

Step 4

Build up a set Ω of the sequences $\mathsf{s}^1, \ldots, \mathsf{s}^r$.

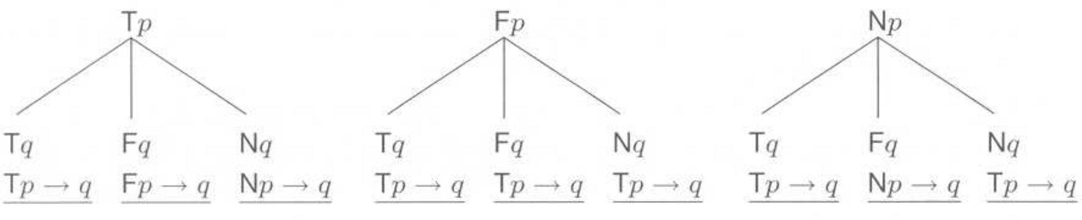
It can be easily checked that Ω is a synthetic tableau for a formula A.

10. *Canonical synthetic tableaux*

Synthetic tableaux built up according to the procedure described in section 9 will be called *canonical*. Let us compare two synthetic tableaux for the formula '$p \rightarrow q$'. The tableau of example 8 is a canonical one, whereas the tableau of example 9 is not:

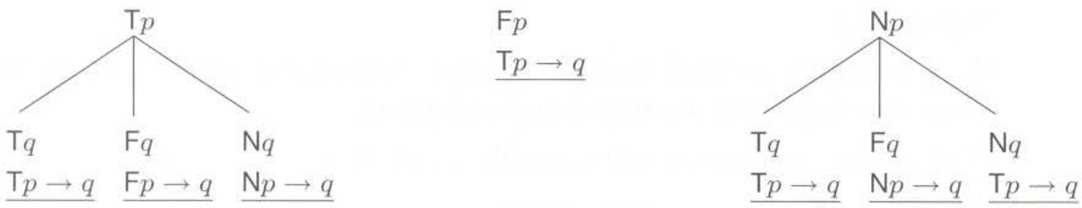
Example 8

A canonical synthetic tableau for the formula '$p \rightarrow q$':



Example 9

A (non-canonical) synthetic tableau for the formula '$p \rightarrow q$':



It is easily seen that the tableau of example 8 contains some inferential steps that are "deductively superfluous": introduction of Tq, Fq, Nq after Fp is not needed in order to obtain T$p \to q$. The procedure described in section 9 enables one to determine the maximal number of synthetic inferences that is sufficient to built up a synthetic tableau for a given formula A: it is equal to 3^k, where k is the number of distinct propositional variables, occurring in A. This procedure does not determine, however, the minimal number of synthetic inferences that are needed to this end. And the method of determining it (as well as the method of determining the minimal length of synthetic inferences) is at the moment the main open problem concerning STM. As can be observed on the basis of the examples 6–9 the problem of the minimal complexity of a synthetic tableau for a given formula should be dealt with in terms of "pragmatics" of a proof rather than in terms of purely structural conditions. Such a heuristics for STM certainly deserves further investigations[6].

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[6] Some preliminary results can be found in [8].

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