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First-order Synthetic Tableaux

RESEARCH REPORT

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First-order Synthetic Tableaux

Research Report

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Our aim is to apply Synthetic Tableaux Method (STM, cf. Urbański (2001) and (2002)) to classical first-order logic (CL). In the formulation of rules of resulting calculus we use a few notions borrowed from certain systems of first-order natural deduction, in particular the system BMV by Bocharov, Markin and Voishvilo (cf. Bolotov et al. (2004)) and the system by Quine (1950).

We will deal here with language of CL with $\neg$, $\wedge$, $\vee$, $\rightarrow$ and both quantifiers as pimitives. We will use individual variables as the only category of names (individual constants and function symbols can be introduced, if needed).

By '$A(x/t)$' we shall mean the result of proper substitution of the term t for x in A.
By '$A(x_{i1}, \ldots, x_{ik})$' we shall mean a formula with $x_{i1}, \ldots, x_{ik}$ as all the distinct free individual variables.
By an *atom* we shall mean an atomic formula or a negation of an atomic formula. If A is an atomic formula, then the atoms A and $\neg A$ will be called *associates* and referred to as *based on* the formula A.

If P is an n-place predicate letter and $t_{i1}, \ldots, t_{ik}$ are terms, then in the expression $P(t_{i1}, \ldots, t_{ik})$, the terms $t_{i1}, \ldots, t_{ik}$ will be referred to as the *scope* of P.

We shall use the following rules:

DN-rule
$A \,/\, \neg\neg A$

CI1-rule	**CI2-rule**	**CR-rule**
$\neg A \,/\, A \rightarrow B$	$B \,/\, A \rightarrow B$	$A, \neg B \,/\, \neg(A \rightarrow B)$
KI-rule	**KR1-rule**	**KR2-rule**
$A, B \,/\, A \wedge B$	$\neg A \,/\, \neg(A \wedge B)$	$\neg B \,/\, \neg(A \wedge B)$
DI1-rule	**DI2-rule**	**DR-rule**
$A \,/\, A \vee B$	$B \,/\, A \vee B$	$\neg A, \neg B \,/\, \neg(A \vee B)$

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GI-rule

$A(x_i / x_j, y_{k1}, \ldots, y_{k2}) \,/\, \forall x_i A(x_i, y_{k1}, \ldots, y_{km})$ — in the conclusion x_j is absolutely restricted and $y_{k1}, \ldots, y_{k2}$ are relatively restricted (x_j restricts $y_{k1}, \ldots, y_{km}$)

EI-rule

$A(x_i / t) \,/\, \exists x_i A(x_i)$ — where t is any term

GN-rule

$\exists x_i \neg A(x_i) \,/\, \neg\forall x_i A(x_i)$

EN-rule

$\forall x_i \neg A(x_i) \,/\, \neg\exists x_i A(x_i)$

RV-rule

renaming of bounded individual variables

RS-rule

substitution of terms for individual variables

In the applications of GI-rule we will be using the notation: $y_{k1}, \ldots, y_{km} < x_i$ to indicate that x_j restricts $y_{k1}, \ldots, y_{km}$. Note, that this relation of restriction is transitive, what means in particular that, if $x < y$ and $y < x$, then y restricts itself.

An intuitive meaning of the notions of restriction and relative restriction is, that because of semantical reasons generalization of a variable x_i limits in a sense (or *restricts*) possible substitution for (free) variables $y_{k1}, \ldots, y_{km}$. Details can be found in Bocharov and Markin (1994).

In the propositional case synthetic tableaux are defined as families of synthetic inferences. A tableau for a formula A consists of synthetic inferences of A or of $\neg A$. In the first-order case, however, we need a notion more general than synthetic inference. The reason is that, because of undecidability of CL, it is possible that an attempt to 'synthesize' certain formula A or its negation fails, that is, that the goal we are aiming at may not be reached. This is why we start with the notion of synthetic derivation for a given formula.

DEFINITION 1

A sequence $\mathbf{s} = s_1, \ldots, s_n$ of formulae is a *synthetic derivation for a formula A* iff:

(1) for any formula s_i of $\mathbf{s}$, s_i is a subformula of A or a negation of a subformula of A;
(2) s_1 is an atom;
(3) for any formula s_g of $\mathbf{s}$, s_g satisfies exactly one of the following conditions:
    (a) s_g is an atom and the associate to it does not appear in $\mathbf{s}$;
    (b) s_g is derivable from a certain set of formulas such that each element of this set occurs in $\mathbf{s}$ before s_g;
(4) no individual variable restricts itself;
(5) for every predicate letter P and for every individual variable x_i, x_i in the scope of P is in $\mathbf{s}$ absolutely restricted at most once.

Conditions 1, 2, 3a and 3b are, respectively, subformula condition, starting condition, introduction of atoms and introduction of compound formulae conditions and they are the same as in the propositional case. The meaning of clauses 4 and 5 will become clear when we present some examples.

Synthetic inferences, in turn, can be interpreted as synthetic derivations that reach their goals:

DEFINITION 2
A *synthetic inference of a formula A* is a synthetic derivation $\mathbf{s} = s_1, \ldots, s_n$ for A such that $s_n = A$.

It is possible, that in two different derivations the very same formula can be derived from different sets of formulae, as in propositional case: $p \rightarrow q$ is a consequence of $\neg p$ and of q as well. In the propositional case this does not cause any troubles. In the first-order case, however, where multiple applications of GI-rule to the very same variable may result in semantical problems, some precautions are needed. In general, we will not allow for multiple generalizations on the same formula within one tableau, with one exception: generalizations on the same formula, being an element of different derivations, will be allowed if the formula will be a point of reconnection of the derivations in question.

DEFINITION 3
Let $\mathbf{s} = s_1, \ldots, s_n$ and $\mathbf{s'} = s'_1, \ldots, s'_m$ be distinct sequences of formulas. **s** and **s'** are *reconnected* iff the following hold:

(1) there exists $e \geq 1$ such that $s_e \neq s'_e$;
(2) there exist h, g ($1<h<n$, $1<g<m$) such that $s_h = s'_g$.

s_h and s'_g are called *points of reconnection* of **s** and **s'**.

Now we are in a position to introduce our main notion:

DEFINITION 4
A family Ω of finite sequences of formulas is a *synthetic tableau for a formula A* iff:

(1) each element of Ω is either a synthetic derivation or a synthetic inference of A or of $\neg A$;
(2) there exists an atomic formula φ such that the first term of every sequence in Ω is an atom based on φ;
(3) for every sequence $\mathbf{s} = s_1, \ldots, s_n$ in Ω the following holds:
  if s_i is an atom, then:
  (a) Ω contains a certain synthetic inference $\mathbf{s}' = s'_1,\ldots,s'_m$ such that s'_i is an associate to s_i and, if $i>1$, then $s'_j = s_j$ for $j = 1,\ldots,i-1$;
  (b) if $i>1$, then for each such a synthetic inference $\mathbf{s}' = s'_1,\ldots,s'_r$ in Ω that $s'_j = s_j$ for $j = 1,\ldots,i-1$, the following holds: $s'_i = s_i$ or s'_i is an associate to s_i;
(4) no individual variable restricts itself in Ω;
(5) for every predicate letter P and for every individual variable x_i, x_i in the scope of P is in Ω absolutely restricted at most once, with the following exception: if **s** and **s'** are reconnected and s_h and s'_g are their points of reconnection, and if s_{h+1} and s'_{g+1} result from applications of GI-rule, then these applications of GI-rule to s_h and s'_g are considered as one.
(6) for every individual variable x_i: if $P_1, \ldots, P_r$ are distinct predicate letters, then x_i can be absolutely restricted in the scope of each of $P_1, \ldots, P_r$ provided that the formulae in which x_i is absolutely restricted are elements of the very same sequence **s** of Ω.

Clauses 1 – 3 are standard within STM-framework. We shall explain the role of the remaining clauses using some examples.

Examples:

1. Synthetic tableau for '$\exists x\forall yP(x,y)$'

$$
\begin{array}{llll}
 & P(x,y) & \neg P(x,y) & \\
x<y & \forall yP(x,y) & \exists y\neg P(x,y) & \\
 & \underline{\exists x\forall yP(x,y)} & \neg\forall yP(x,y) & \\
 & & \forall x\neg\forall yP(x,y) & <x \\
 & & \underline{\neg\exists x\forall yP(x,y)} &
\end{array}
$$

In the inference starting with $\neg P(x,y)$ the variable x is absolutely restricted but it does not relatively restrict any other variable. Note, that if y were free in $\neg\forall yP(x,y)$ it would not be possible to use GI-rule here as in this case we would have $x<y$, $y<x$ and, by transitivity, $x<x$ (cf. clause 4 of definition 4) – as in example 3 below.

2. Synthetic tableau for '$\exists x\forall yP(x,y) \rightarrow \forall y\exists xP(x,y)$'

$$
\begin{array}{llll}
 & P(x,y) & \neg P(x,y) & \\
 & \exists xP(x,y) & \exists y\neg P(x,y) & \\
<y & \forall y\exists xP(x,y) & \neg\forall yP(x,y) & \\
 & \underline{\exists x\forall yP(x,y) \rightarrow \forall y\exists xP(x,y)} & \forall x\neg\forall yP(x,y) & <x \\
 & & \neg\exists x\forall yP(x,y) & \\
 & & \underline{\exists x\forall yP(x,y) \rightarrow \forall y\exists xP(x,y)} &
\end{array}
$$

3. Synthetic tableau for '$\forall y\exists xP(x,y) \rightarrow \exists x\forall yP(x,y)$'

$$
\begin{array}{llll}
 & P(x,y) & \neg P(x,y) & \\
x<y & \forall xP(x,y) & \forall x\neg P(x,y) & x<y \\
 & \exists x\forall yP(x,y) & & \\
 & \underline{\exists x\forall yP(x,y) \rightarrow \forall y\exists xP(x,y)} & &
\end{array}
$$

In the inference on the left y restricts x and the formula $\exists x\forall yP(x,y) \rightarrow \forall y\exists xP(x,y)$ is derived. In fact, this is the only synthetic inference in the above tableau. On the right we have an application of GI-rule which leads to self-restriction of x. Strictly speaking, we should not interpret the formula $\forall x\neg P(x,y)$ as an element of the tableau (cf. definition 4, clause 4). We write it down, however, in order to show why it is not possible to derive neither the formula in question nor its negation here.

4. Synthetic tableau for ‘$\exists xR(x) \to \forall xR(x)$’

$$\begin{array}{llll} & R(x) & \neg R(x) & \\ <x & \forall xR(x) & \forall x\neg R(x) & <x \\ & \underline{\exists xR(x) \to \forall xR(x)} & & \end{array}$$

In this tableau x is absolutely restricted twice and, as $R(x)$ and $\neg R(x)$ are not points of reconnection, this violates clause 5 of definition 4 (an analogous clause can be found in Quine (1950)).

5. Synthetic tableau for ‘$\forall xR(x) \to \exists xR(x)$’

$$\begin{array}{ll} R(x) & \neg R(x) \\ \exists xR(x) & \exists x\neg R(x) \\ \underline{\forall xR(x) \to \exists xR(x)} & \neg\forall xR(x) \\ & \underline{\forall xR(x) \to \exists xR(x)} \end{array}$$

6. Synthetic tableau for ‘$\forall xR(x) \vee \forall xQ(x) \to \forall x(R(x)\vee Q(x))$’

$$\begin{array}{lllll} & R(x) & & \neg R(x) & \\ & R(x)\vee Q(x) & & \exists x\neg R(x) & \\ <x & \forall x(R(x)\vee Q(x)) & & \neg\forall xR(x) & \\ & \underline{\forall xR(x) \vee \forall xQ(x) \to \forall x(R(x)\vee Q(x))} & & & \\ & & & Q(x) & \neg Q(x) \\ & & & R(x)\vee Q(x) & \exists x\neg Q(x) \\ & & <x & \forall x(R(x)\vee Q(x)) & \neg\forall xQ(x) \\ & & & \underline{\forall xR(x) \vee \forall xQ(x) \to \forall x(R(x)\vee Q(x))} & \neg(\forall xR(x)\vee\forall xQ(x)) \\ & & & & \underline{\forall xR(x) \vee \forall xQ(x) \to \forall x(R(x)\vee Q(x))} \end{array}$$

Here x is absolutely restricted twice but the two occurences of $R(x)\vee Q(x)$ are points of reconnections of the inferences in question. Thus, unlike Quine (1950) and Bocharov and Markin (1994), we in fact consider some applications of GI-rule as one and the same. This is one of two exceptions to the rule, that an individual variable can be absolutely restricted in a tableau at most once.

7. Synthetic inference for ‘$\forall x(R(x)\wedge Q(x)) \to \forall xR(x) \wedge \forall xQ(x)$’

$$\begin{array}{lllll} & & & R(x) & \neg R(x) \\ & & <x & \forall xR(x) & \neg(R(x)\wedge Q(x)) \\ & & & & \exists x\neg(R(x)\wedge Q(x)) \\ & & & & \neg\forall x(R(x)\wedge Q(x)) \\ & Q(x) & & \neg Q(x) & \underline{\forall x(R(x)\wedge Q(x)) \to \forall xR(x) \wedge \forall xQ(x)} \\ <x & \forall xQ(x) & & \neg(R(x)\wedge Q(x)) & \\ & \forall xR(x) \wedge \forall xQ(x) & & \exists x\neg(R(x)\wedge Q(x)) & \\ & \underline{\forall x(R(x)\wedge Q(x)) \to \forall xR(x) \wedge \forall xQ(x)} & & \neg\forall x(R(x)\wedge Q(x)) & \\ & & & \underline{\forall x(R(x)\wedge Q(x)) \to \forall xR(x) \wedge \forall xQ(x)} & \end{array}$$

In this tableau x is restricted twice in the way that is allowed by clause 6 of definition 4: x is absolutely restricted in scope of distinct predicate letters but on the very same path of the tableau (this is the second exception to the rule: one variable – one tableau – at most one absolute restriction).

In order to introduce the notion of proof we need two more definitions:

DEFINITION 5
A synthetic inference **s** of a formula A is *finished* iff no individual variable absolutely restricted in **s** is free in A.

DEFINITION 6
A synthetic tableau Ω for a formula A is *finished* iff all the elements of Ω are finished synthetic inferences of A or of $\neg A$.

Semantical justification for these concepts can be found in Quine (1950) and, to some extent, in Bocharov and Markin (1994).

DEFINITION 7
A *proof* of a formula A is a finished synthetic tableau Ω for A such that each path of Ω leads to A.

THEOREM
A formula A is valid iff there exists a proof of A.

We suppose that the proof of soundness of the calculus could be similar to the proof of semantical consistency of the system BMV by Shangin (cf. Shangin (2004)). On the other hand, the most efficient way to prove completeness seems to be the method of interpretation. Most promising here are first-order calculi of Socratic proofs (cf. Wiśniewski, Shangin (forthcoming)).

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